Relevant Variants of Intuitionistic Logic

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1 Introduction

Our aim is to construct a number of systems which formalize relevant entailment for formulas of language, obtained from propositional intuitionistic language by means of replacement of intuitionistic implication by intensional E-type implication '$\rightarrow$'. Thus, we will build up some relevant constructive systems which would be analogous to classical relevance logic E.

We shall start from the positive relevance logic $E+$ (see [1, 10]) and its semantics. Let us recall how this semantics is formulated. A model structure for E^+ ($E+$-m.s.) is a triple $\langle O, S, R\rangle$, where S is a nonempty set of 'possible worlds' ('set-ups'), $O \in S$, and R is a ternary relation on S, such that the following postulates hold for all a, b, c, d, e in S with quantifiers ranging over S:

PROPERTY 1.1

1. $ROaa$
2. $RaOa$
3. ($Rabc$ and $Rcde$) $\Rightarrow \exists x$($Radx$ and $Rbxe$)
4. $Rabc \Rightarrow \exists x$($Rabs$ and $Rxbc$)
5. ($ROad$ and $Rdbc$) $\Rightarrow Rabc$.

Let $\mathbf{T}A/a$ mean 'formula A is true in the world a'. Then the following condition ('monotonicity') must be held when we assign truth-value to each sentential variable p_i of $L(E)+$ at each world a of S:

DEFINITION 1.2
$ROab$ and $\mathbf{T}p_i/a \Rightarrow \mathbf{T}p_i/b$

Truth-conditions for compound formulas are determined by means of the following definitions:

DEFINITION 1.3

1. $\mathbf{T}A\&B/a \Leftrightarrow \mathbf{T}A/a$ and $\mathbf{T}B/a$
2. $\mathbf{T}A \vee B/a \Leftrightarrow \mathbf{T}A/a$ or $\mathbf{T}B/a$
3. $\mathbf{T}A \rightarrow B/a \Leftrightarrow \forall b \forall c(Rabc \rightarrow (\mathbf{T}A/b \rightarrow \mathbf{T}b/c))$.

A formula A is verified in $E+$-m.s. $\langle O, S, R\rangle$ on a given definition of truth-condition for propositional variables just in case $\mathbf{T}A/O$; definitions of validity in a given $E+$-m.s. and $E+$-validity are standard as well.

So, we take the system $E+$ as a positive basis of relevance intuitionistic systems. It remains only to add to $E+$ axiom schemes for negation, and supplement a given semantics accordingly.

2 A minimal system of relevant entailment

At first we shall use *as a heuristic method* a well-known technique with a sentential constant f [3]. Then we shall formulate some systems with negation as a primitive symbol. The language $L(E)f$ is obtained from the language $L(E)+$ by adding a constant f to the alphabet of the latter. Then we can have the calculus ME_f which is determined by axiom schemes and rules of the system $E+$. Negation can be defined in ME_f as usual with

DEFINITION 2.1
$\neg A \Leftrightarrow A \rightarrow f$.

An ME- model structure (ME-m.s.) is $\langle O, S, N, R\rangle$, where S is a nonempty set, $O \in S$, $N \subseteq S$ (N can be empty), and R is a ternary relation on S, satisfying properties 1.1.1–1.1.5.

Condition 1.2 continues to hold. Besides that the following condition is taken for ME-m.s.:

DEFINITION 2.2
$ROab$ and $a \in N \Rightarrow b \in N$.

To definitions 1.3.1–3 we add

DEFINITION 2.3
$\mathbf{T}f/a \Leftrightarrow a \in N$.

The intuitive understanding of the above modification of semantics for $E+$ is quite transparent. A set of 'possible worlds' S can be understood as a set of theoretical constructions. In the present case, as we deal with intuitionistic logic, it can be a set of possible intuitionistic theories. That is, a theoretical statement belongs to the world a if this statement is intuitionistically proved. We distinguish from the set S a subset of contradictory theories N; i.e. a theory a belongs to N if this theory is a contradictory one; in other words, there exists a contradictory statement that belongs to the theory a. ($1 = 2$ can be considered as an example of such a statement). In such a case the constant f just represents a contradictory statement. Thus, the definition 2.3 has the following sense; the constant f is true in a world a if this world is contradictory (i.e. it is a contradictory theory). Notice that including contradictory worlds in relevant model structures is not anything unnatural. On the contrary, one of the features of semantics for relevant systems is that here a belonging of contradictory worlds to model structures is allowed (see on this matter in [9, p134] [12, 13], and elsewhere).

In the sequel the following ME_f-theorems will play an important role:

THEOREM 2.4
1. $(A \rightarrow B) \rightarrow ((B \rightarrow f) \rightarrow (A \rightarrow f))$;
2. $(A \rightarrow (A \rightarrow f)) \rightarrow (A \rightarrow f)$;
3. $(A \rightarrow B) \rightarrow ((A \rightarrow (B \rightarrow f)) \rightarrow (A \rightarrow f))$;
4. $(A \rightarrow f) \rightarrow (((A \rightarrow f) \rightarrow B) \rightarrow B)$
5. $(A \rightarrow B) \rightarrow (((A \rightarrow B) \rightarrow f) \rightarrow f)$;
6. $(A \rightarrow f) \rightarrow (((A \rightarrow f) \rightarrow f) \rightarrow f)$;
7. $(A \rightarrow ((B \rightarrow f) \rightarrow f)) \rightarrow ((B \rightarrow f) \rightarrow (A \rightarrow f))$.

These theorems explain some essential properties of minimal relevant negation introduced by definition 2.1 that can be obtained from 2.4.1–2.4.7 by writing $A \to f$ instead of $\neg A$, must be theorems of the system ME we are going to formulate below. In this system a minimal relevant E(-type) negation is formalized by means of special axiom schemes for '$\neg$'. It is important to take into account that the following formulas are *not* derivable in ME_f: $A \to ((A \to f) \to f)$ and $(A \to (B \to f)) \to (B \to (A \to f))$. The problem is that the corresponding formulas $A \to ((A \to B) \to B)$ and $(A \to (B \to C)) \to (B \to (A \to C))$ are not theorems of $E+$ (and E),[1] and there is no special axiom for f in ME_f. Thus, when formulating system ME, it is necessary to take precaution. We get a propositional calculus ME if we add to axiom schemes of $E+$ the following schemes for negation:

$ME13.$ $(A \to \neg A) \to \neg A$ $\quad$ $ME15.$ $\neg A \to ((\neg A \to B) \to B)$

$ME14.$ $(A \to B) \to (\neg B \to \neg A)$ $\quad$ $ME16.$ $(A \to B) \to \neg\neg(A \to B)$.

Perhaps, earnest explanation concerning schemata $ME15$ is necessary. This scheme is worthy of detailed consideration. We include it in the system ME, because the formula in theorem 2.4.4 is provable in ME_f. Apparently it can be seem rather questionable and against the spirit of the system E (as we claim that ME is a system of E-type) that $ME15$ belongs to system ME. $ME15$ is not provable in the system E (unlike system R!). But we *had to* take $ME15$ as an axiom scheme, proceeding from features of system $E+$ and from definition 2.1. Thus, in ME *Permutation* is allowed not only for formulas of the form $A \to B$ (as it does in E), but also for formulas of the form $\neg A$. Formulas $(A \to (\neg B \to C)) \to (\neg B \to (A \to C))$ and $\neg A \to ((\neg A \to \neg A) \to \neg A)$ ($\neg A \to N \neg A$ - a true negative statement is necessary) are theorems of ME. But this fact must not disencourage at all, because one could foresee it! Indeed, in ME we deal with *intuitionistic-type* negation. And such a negation differs essentially from a classical one. The difference is bound with a specificity of intuitionistic statements (constructivity), and with aspiration of intuitionism so that every negative statement would also be constructive. Therefore statement $\neg A$ is understood in intuitionism as 'A is refuted intuitionistically (constructive)', i.e. in intuitionism $\neg A$ is considered as true if an assumption that A is true leads to a contradiction. (see, e.g. in [2, p. 98]) Thus, propositions with intuitionistic negation as a main connective are not statements of factual nature (contingent statements). In intuitionism, to every statement of form $\neg A$ some inference is associated (an inference of a contradiction from A). In other words, here it is possible to assert $\neg A$ (where '$\neg$' is intuitionistic-type negation), only when a contradiction *follows logically* from A (definition 2.1 asserts this fact syntactically). In that case $ME15$ simply spreads principles laid down in the $E+$-axiom $(A \to ((B \to C) \to D)) \to\to ((B \to C) \to (A \to D))$ (*Restricted Permutation*) on negative intuitionistic propositions.

We get a semantics for ME if we change definition 2.3 by

DEFINITION 2.5

$\mathbf{T}\neg A/a \Leftrightarrow \forall b \forall c (Rabc \Rightarrow (\mathbf{T}A/b \Rightarrow C \in N))$

This definition literally reproduces an intuitionistic informal understanding of negation. Let us remember that N is a set of contradictory 'worlds'. So, $\neg A$ is true if, when we suppose that A is true, then we are led to a contradiction.

[1] But they are theorems of system $R+$ (and R)! As it is well known, R differs from E in that *Unrestricted Permutation* is acceptable in R.

In the framework of ME-m.s. one can define in the usual way a relation of relevant logical entailment for all formulas of ME:

DEFINITION 2.6
$A \models_{me} B \Leftrightarrow Sa \in S(\mathbf{T}A/a \rightarrow \mathbf{T}B/a)$.

The following theorems can now be proved for ME:

THEOREM 2.7
$ROab$ and $\mathbf{T}A/a \rightarrow \mathbf{T}A/b$.

THEOREM 2.8
$\models_{ME} A \rightarrow B \Leftrightarrow A \models B_{ME}$

THEOREM 2.9
$Raaa$.

THEOREM 2.10
$RaOb$ and $\mathbf{T}A \rightarrow B/a \rightarrow \mathbf{T}A - B/b$.

THEOREM 2.11
$RaOb$ and $\mathbf{T}\neg A/a \rightarrow \mathbf{T}\neg A/b$.

Consistency and completeness can also be proved. Let us sketch some focal points of the completeness proof. An *intensional ME-theory* (in the future *intensional theory*) is a set x of formulas of the language $L(ME)$, satisfying the following conditions:

(i) $A \in x$ and $\vdash_{ME} A \rightarrow B \Rightarrow B \in X$; (ii) $A \in x$ and $B \in x \Rightarrow A \wedge B \in x$.

An intensional theory is *prime* iff $A \vee B \in x \Rightarrow A \in x$ or $B \in x$. An intentional theory x is *contradictory* if there exists an ME-theorem (L) such that $\neg L \in x$. Let PIT is the set of all prime intensional theories (including contradictory ones). For all x, y, z from PIT we define a relation R' as follows: $R'xyz \Leftrightarrow (A \rightarrow B \in x \Rightarrow (A \in y \Rightarrow B \in z))$. Let N' be the set of all contradictory prime intensional theories. By means of ME_t we mark the set of all theorems os the system ME. Then we call $\langle ME_t, PIT, N', R' \rangle$ a *canonical structure*.

LEMMA 2.12
The canonical structure is ME-m.s.

Now, let us define the canonical definition of truth-condition for propositional variables as follows: for all p_i, for every $x \in PIT\mathbf{T}p_i/x \Leftrightarrow p_i \in x$.

LEMMA 2.13
For every formula A of the language $L(ME)$ and for every $x \in PIT$, $\mathbf{T}A/x \Leftrightarrow A \in x$.

THEOREM 2.14
If $\models_{me} A$, then $\vdash_{me} A$.

PROOF. Let $\models_{me} A$. That is A is valid in all $ME - m.s$. By lemma 2.12 it is valid in canonical structure. Hence A is verified in ME_t by all definitions of truth-condition for propositional variables. Let us consider the canonical definition of truth-condition for propositional variables. By lemma 2.13 we have $A \in ME_t$, i.e. A is a theorem of the system ME. ∎

Now we shall consider some features of our semantics as compared with other relevant semantics built according to the so-called Australian plan (i.e. semantics of Routley–Meyer [9, 10] and Maximova [4]). A trait of the semantics on the Australian plan consists in the use of (side by side with a ternary relation R) operation $*$ which is defined on a set of possible worlds. By means of this operation the truth condition for negation is defined ($\mathbf{T}\neg A/a \Leftrightarrow$ not $\mathbf{T}A/a*$). The presence of $*$ in Australian semantics caused (and still causes) numerous hot discussions. Many research-workers are not content with such a definition of truth-condition for negative propositions. For example, prof. Voishvillo writes that this definition 'hides the real sense of negation operator in E. It gained an impression that negation in E is not a classical one.' (see [13, p. 114]) In this connection some attempts to do in semantics without the 'star' (see e.g. [8]) have been undertaken. As opposed to such a critical attitude to 'star', R. Meyer and E. Martin [7] come out in defence of such a, as they consider, 'valuable notion'. They write: '$*$ seems to us an important theoretical ingredient in Relevant semantical analysis. One may re-describe its effect – one may even define it in terms of other notions – but one does not escape it.' ([7, p. 310]) We shall not enter into discussion about whether or not the 'star' is necessary, and, if it is necessary, whether it is a semantical *ad hoc* or one can intuitively explain it in some natural way. Let us only note that our semantics for ME (see also the semantics for 'constructive R' in [5]) demonstrates clearly that all the problems with 'star' arise in Relevant semantics just in case when we want to have a *classical-type* negation. But, if a question is about an *intuitionistic-type* negation, it turns out that there is no need to have an $*$ operator at all. In particular, by means of definition 2.5 the truth-condition for minimal negation of the system ME is defined quite naturally and without using such a 'suspicious' and vague entity as 'star'. In fact, it is doubtful whether one can bring claims to the definition 2.5 as he does to the 'starry' definition for negation in semantics of Routley–Meyer and Maximova (that it is unnatural and hides the sense of negation operator). Thus, in relevant semantical analysis of intuitionistic logic at least one 'vague place' of model structures of relevant classical logics is missing. It seems to us that this fact is quite eloquent. Perhaps it is a supplementary argument in favour of an opinion that 'relevant implication is rather more intuitionistic than classical' ([11, p. 167]).

We conclude this section attention to the fact that although formulas

$$A \rightarrow \neg\neg A \tag{2.1}$$

and

$$(A \rightarrow \neg B) \rightarrow (B \rightarrow \neg A) \tag{2.2}$$

are not theorems of ME, their weakened variants

$$\neg A \rightarrow \neg\neg\neg A \tag{2.3}$$

and

$$(A \rightarrow \neg\neg\neg B) \longrightarrow (\neg B \rightarrow \neg A) \tag{2.4}$$

are provable in this system.

3 Relevant intuitionistic system IE

As it was mentioned above, a number of formulas, which we would wish to have as theorems, were not provable in ME (e.g. 2.1 and 2.2). The reason for this is that *Unrestricted Permutation* is not permissible in the system $E+$ (neither is it allowed in E). Here it is lawful to rearrange only formulas which are statements about entailment (of forms $A \to B$) – *Restricted Permutation.* Substituting the constant f instead of C and D in this axiom, we have got 2.4.7 as a theorem of ME_f – so, 2.3 and 2.4 as theorems of ME. But if we want to have 2.1 and 2.2 in our system, we have to take one of them as an axiom.

We get f-formulation of the system of relevant intuitionistic entailment – IE_f – if we add to axioms of $E+$ the following scheme:

$$fE13. \quad A \to ((A \to f) \to f).$$

We get the system IE with negation as a primitive symbol, if we replace the scheme $ME16$ by a stronger one:

$$IE16. \quad A \to \neg A.$$

Semantics for IE_f and IE

IE-m.s. is obtained from ME-m.s., by adding to postulates 1.1.1–1.1.5 the following postulate:

p6.c $\notin N$ and $Rabc \Rightarrow Rbac$.

All the other postulates and definitions we leave without any change.

Now, as to Permutation, the system IE is even more 'liberal' than ME. It turns out that in IE (as well as in IE_f) permutation of antecedents of implicative formulas is permissible not only when these antecedents has the form $A \to B$ or $\neg A$, but also when the last consequent represents a false proposition (constant f). The presence of 2.1 (more transparent – $(A \to (B \to f)) \to (B \to (A \to f))$) in $IE(IE_f)$ as a theorem just points out this fact. By the way, in IE formulas which directly express this fact are provable, namely $A \to ((A \to \neg L) \to \neg L) <$ equivalently $(A \to (B \to \neg L)) \to (B \to (A \to \neg L))$, where L is a theorem of IE which has the form $C \to D$ or $\neg C$. Indeed, we have the following sketch of inference:

1.	$(L \to \neg A) \to (L \to \neg A)$	$E1$
2.	$(L \to \neg A) \to (L \to \neg\neg\neg A)$	from 1, using $IE16$
3.	$L \to ((L \to \neg A) \to \neg\neg\neg A)$	from 2, using *Restricted Permutation*
4.	$L \to ((\neg\neg A) \to \neg(L \to \neg A))$	from 3, using 2.1
5.	$\neg\neg A \to (L \to \neg(L \to \neg A))$	from 4, using $ME15$.
6.	$\neg\neg A \to ((L \to \neg A) \to \neg L)$	from 5, using 2.2.
7.	$\neg\neg A \to ((A \to \neg L) \to \neg L)$	from 6, using the fact that $L \to \neg A$ and $A \to \neg L$ are equal
8.	$A \to ((A \to \neg L) \to \neg L)$	from 7, using $IE16$.

Thus, construction of relevant variants of intuitionistic logic has led to some interesting results. Notwithstanding the fact that the set of theorems of intuitionistic propositional calculus is a subset of the set of classical propositional calculus, the analogous assertion about IE (as well as ME) and E is wrong. That is, we cannot obtain a relevant variant (E-type) of intuitionistic logic, simply by excluding $\neg\neg A \to A$ from the list of axioms of E. It is necessary to add $ME15$, and this requirement is conditioned by the features of intuitionistic negation and implication of E-type.

Theorems 2.7–2.11 hold for IE, as well as results about consistency and completeness (proofs *mutatis mutandis*).

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