
The Trilattice of Constructive Truth Values

YAROSLAV SHRAMKO, *Department of Philosophy, State Pedagogical University, 50086 Kryvyi Rih, Ukraine.*
E-mail: kff@kpi.dp.ua

J. MICHAEL DUNN, *School of Informatics, and Departments of Philosophy and Computer Science, Indiana University, Bloomington, IN 47405, USA.*
E-mail: dunn@indiana.edu

TATSUTOSHI TAKENAKA, *Information Storage Division, The 4th Department, Japan Patent Office, 3-4-3 Kasumigaseki, Chiyoda-ku, Tokyo 100-8915, Japan.*
E-mail: takenaka-tatsutoshi@jpo.go.jp

Abstract

We introduce an abstract algebraic structure — a lattice defined on a generalized truth value space of constructive logic. For background one can refer to the idea of ‘under-determined’ and ‘over-determined’ valuations (Dunn), a ‘useful four-valued logic’ (Belnap), and the notion of a bilattice (Ginsberg). We consider within one general framework the notions of constructive truth and constructive falsity, as well as the notions of non-constructive truth and non-constructive falsity. All possible combinations of the basic truth values give rise to an interesting ‘16-valued logic’. It appears that these 16 truth values constitute what we call a *trilattice* — a natural mathematical structure with three partial orderings that represent respectively an increase in information, truth and constructivity. The presentation of the paper is essentially *conceptual*: the stress is laid on introducing new concepts and structures as well as on their general interpretation.

Keywords: Under-determined valuation, over-determined valuation, truth value space, bilattice, trilattice, constructive truth values, many-valued logic.

1 General background. Multivaluations, four-valued logic and bilattices

In modern symbolic logic it is by no means unusual to allow sentences to be sometimes *neither* true *nor* false. This idea goes back to Łukasiewicz, and has been implemented in various systems of many-valued logic, partial logic and elsewhere. But there is also a ‘dual’ idea that until recently had not gained many supporters among the ‘logical community’. The idea is that some sentences can occasionally be viewed as having *both* of these truth values. Nevertheless, in the last couple of decades various researchers have repeatedly noticed that it is sometimes quite *useful* to value sentences as both true and false. This view has found, recently, numerous fruitful applications in paraconsistent logics, logics of belief and in computer science.

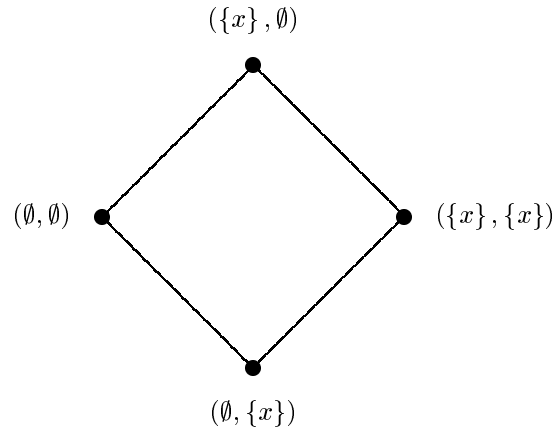


FIGURE 1. De Morgan lattice of proposition surrogates

Unfortunately, it is not so often recognized that this approach was carefully elaborated in the 1960s and 1970s by investigations of implication and entailment within *relevance logic*. Moreover, in these investigations both ideas mentioned above were brought together and combined by considering the so-called ‘under-determined’ and ‘over-determined’ valuations — side by side with the usual ones.

The point that can be found in [11] and [12] (see also in [4] and [13]) is that one can generalize the ordinary notion of a truth-value function, and consider a ‘valuation’ to be a function not from sentences to the *elements* of the set of two truth values $\{\mathbf{T}, \mathbf{F}\}$, but from sentences to the *subsets* of this set. In this way truth and falsity no longer satisfy the classical meta-principle of *Bi-unique Valence*, according to which every sentence has precisely one of these values. That is, a sentence *is allowed* to be neither true nor false, as well as both true and false simultaneously. In [12, p. 156], this kind of valuation has been called a ‘relevance valuation’ with a specific goal in mind to provide a semantics for a relevant entailment relation. [11] (Chapter IX) prefers to speak of the so-called ‘aboutness valuation’, starting from the idea that every sentence is ‘about certain topics’. Thus, we can assign to each proposition p a pair (X_1, X_2) (called a ‘proposition surrogate’, see also [12, p. 161]), where X_1 and X_2 are subsets of a ‘universe of discourse’ X that are thought, respectively as ‘the topics that p gives definite information about and the topics that $\sim p$ gives definite information about’ [11, p. 126]. If the universe of discourse consists of a single topic x , the usual (‘normal’) valuations can be represented as $(\{x\}, \emptyset)$ and $(\emptyset, \{x\})$ — for truth and for falsity respectively, and under-determined and over-determined valuations as (\emptyset, \emptyset) and $(\{x\}, \{x\})$. It appears that these valuations form a lattice. One finds this lattice in [11, p. 129], which is called there ‘De Morgan lattice of proposition surrogates’. We present this lattice in Figure 1 in an ‘upside-down’ form for the sake of uniformity with further exposition.

Let us generally call this kind of valuation a *multivaluation*, and the corresponding semantic construction a *multivaluational semantics*. The usual (classical) valuations appear to be then some particular cases of a multivaluation, namely when a multivaluation picks up precisely *one* element from the initial set of truth values.

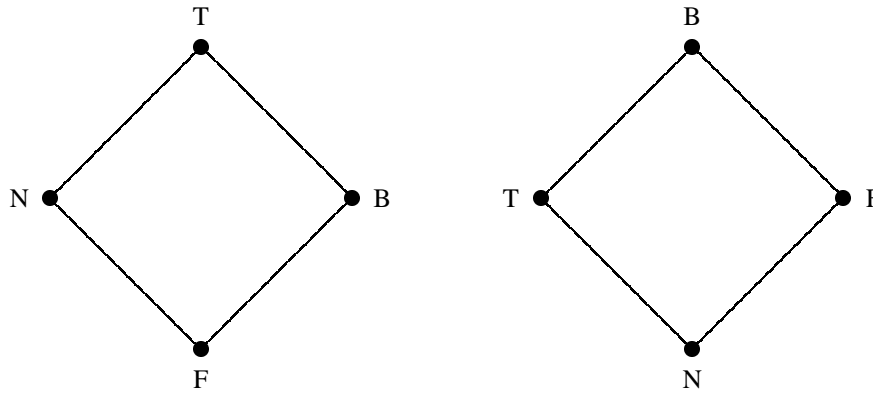


FIGURE 2. Logical lattice L4 and approximation lattice A4

As already noticed, introducing the ‘under-determined’ and ‘over-determined’ valuations can be *useful* in various aspects. For example, [12] employs these valuations to construct an ‘intuitive semantics’ for first-degree entailment that invalidates the so-called ‘Paradoxes of Consequence’. This semantics naturally incorporates a rather intuitive motivation for the ‘non-normal’ valuations, which is provided in terms of (epistemic) abstract ‘situations’ (see [12, pp. 155–157]).

Another important field of application comes from computer science and logic programming. It was Belnap who proposed a highly heuristic ‘computerized’ interpretation of the abstract epistemic situations involving the fact that computers often have to work with incomplete and/or inconsistent databases, and that we might wish that a computer still perform with some degree of reliability using such data [5, 6]. He also explicitly presented the empty and the ‘over-complete’ subsets of the set $\{T, F\}$ as new *truth values* speaking thus of a (‘useful’) *four-valued* logic.¹ If we understand a value as information that ‘is told’ to a computer, we have to take into account, side by side with the ‘usual’ (or ‘normal’) situations when a computer is told either truth or falsehood, also the situations when a computer is told *both* truth and falsehood, as well as *none* of them. Thus, we arrive at the following four ‘told values’ that correspond to the above mentioned multivaluations:

- $T = \{T\}$ — ‘plain’ truth;
- $F = \{F\}$ — ‘plain’ falsehood;
- $B = \{T, F\}$ — *both* truth and falsehood;
- $N = \{\}$ — *neither* truth nor falsehood.

We will call these truth values, and more generally the truth values obtained by the ‘multi-valuational approach’, *generalized truth values*. Now the lattice from Figure 1 can be explicitly represented as a lattice of the four generalized truth values as in Figure 2, left. Belnap [5, p. 14] calls it ‘logical lattice **L4**.’ This lattice is ‘logical’ because the ordering on it is in effect a *logical order* with the usual truth-functional conjunction and disjunction as meet and join respectively.

¹The idea of the four truth values has been also expressed by D. Scott (see [41, p. 170]).

We may say that a sentence is ‘at least true’ if it contains **T** in its ‘told value’. Then one can define a *relation of entailment* as follows: A entails B iff B is *at least true* whenever A is. It appears that this relation is *relevant* in that it invalidates the above mentioned ‘Paradoxes of Consequence’. The ‘four-valued semantics’ is axiomatized by the system \mathbf{R}_{fde} for the ‘first degree entailments’ of the relevance logic \mathbf{R} (as well as \mathbf{E}) (see, [15]). This system is exactly the system of ‘tautological entailments’ from [3, Chapter III].

Belnap’s papers present also another lattice based on the same elements. Belnap calls it the ‘approximation lattice **A4**’ (Figure 2, right) because its ordering can be naturally explicated as ‘approximates the information in’. The idea of this lattice goes back to Scott (see, e.g. [41]) who considers various examples of an approximation order. Belnap remarks that **N** is the bottom of **A4** because it gives no information at all, whereas **B** is at the top because it gives too much (indeed inconsistent) information.

The next step, which is implicit in Belnap, is to combine these two lattices and to treat them from a ‘joint perspective’. This step has been explicitly done by Ginsberg in [21] and [22] who introduced the notion of a *bilattice*. Roughly the bilattice is a non-empty set with *two* partial orderings \leq_i and \leq_t , each constituting a complete lattice on this set. Informally, \leq_i represents an increase in information² and \leq_t - an increase in truth. The combination of the lattices **L4** and **A4** forms the simplest non-trivial bilattice which we call here $FOUR_2$. This bilattice³ can be presented by means of a double Hasse diagram with an explicit use of a two-dimensional coordinate plane, where the horizontal axis stands for a truth ordering and the vertical axis represents an information ordering (Figure 3).

Bilattices have been extensively studied by Fitting [17, 18], Arieli and Avron [2], and other authors [42, 19, 37, 38]. Figure 4 presents some typical bilattices very well known from the literature (we change as appropriate the labels for some truth values in accordance with terminology adopted in the present paper): $SEVEN_2$ (introduced in [21] as a suitable tool for non-monotonic reasoning in default logic) and $NINE_2$ [2]. The values **dt** and **df** both in $SEVEN_2$ and $NINE_2$ mean ‘true by default’ and ‘false by default’ respectively. Unfortunately Arieli and Avron provide no explanations about the informal meaning of **ot** and **of**.

Fitting points out that a bilattice can be thought of as a *generalized truth value space* [17, p. 225]. We will shortly compare the truth value space represented by $FOUR_2$ with the ones represented by $SEVEN_2$ and $NINE_2$. Some authors are inclined not to pay much attention to (or even disregard) the following crucial difference between them. While the truth values in $FOUR_2$ arise as a natural result of creating ‘multivalues’ from a set of *initial* truth values (in this case from just two elements), the truth values in $SEVEN_2$ and $NINE_2$ seem to be chosen rather arbitrary, mainly on the basis of some not quite exact intuitive observations. $FOUR_2$ is obtained by using a certain uniform method: one takes an initial set of truth values

²We will call it an *information ordering*. This label is more instructive than Scott’s ‘neutral’ notion of approximation which is far too general. Note, that Ginsberg uses the subscript k instead of i and speaks of a ‘knowledge ordering’, what is not quite accurate from a philosophical point of view. Although it is common among computer scientists to use the term ‘knowledge’ in a very general sense, philosophers usually mean by knowledge only *true* beliefs. Therefore it is more appropriate to speak cautiously just of information and information ordering rather than of knowledge.

³The usual name for this bilattice is just *FOUR*. We add to this name a subscript 2, as we will generally do in case of bilattices. We will keep using this style of notation, when we later introduce the notion of trilattice, and also when we deal with some intermediate structures and possible extensions. The point is that sometimes *the same* number of elements can produce quite different structures (e.g. monolattice, bilattice, trilattice, etc.). Thus, pointing out just the amount of elements says not much of the nature of structure we are dealing with in a particular case. Using an appropriate subscript makes this clear.

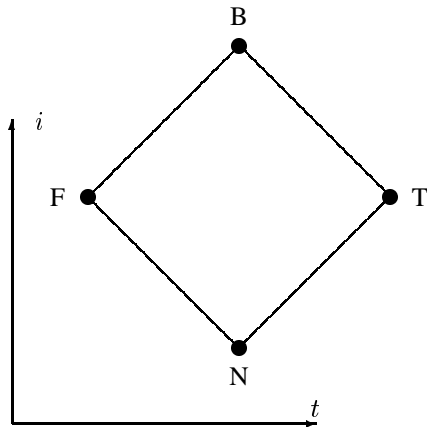


FIGURE 3. Bilattice $FOUR_2$

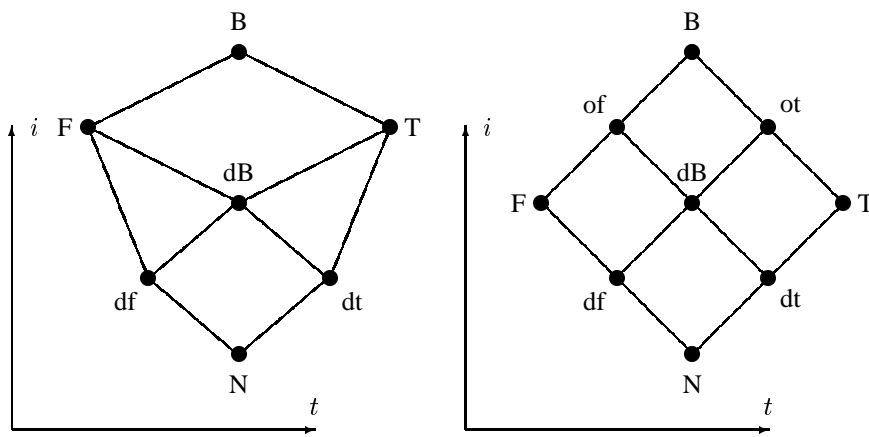


FIGURE 4. Bilattices $SEVEN_2$ and $NINE_2$

(which can be thought of as the *base* of the resulting truth value space), and then one takes *all* possible combinations of the elements of this set (multivalues). The resulting truth value space is in a sense ‘complete’ (or *perfect*). This is not the case with $SEVEN_2$ or $NINE_2$. Indeed, one can well ask, why we consider a truth value which is the combination of **T** and **F**, but ignore the combination of **dt** and **F**? This smell of *ad hocness* becomes stronger when we consider some specific features of the corresponding bilattices. For example, one can doubt that **dB** is of the same degree of information as **F** and **T** (as in $NINE_2$), or one can doubt that disjunction of **B** and **dB** should produce just **T** (as in $SEVEN_2$). Likewise one can ask, why in $NINE_2$ **dB** occurs on the same ‘informational level’ as **F** and **T**, and in $SEVEN_2$ it is situated on some ‘lower’ level.

Therefore, we consider $FOUR_2$ a *paradigmatic* bilattice based on just two initial elements. However, reflecting upon the nature of this space, one can notice that the initial truth values that form its ‘base’ are *classical* (as well as the ‘normal’ truth and falsity in $SEVEN_2$ and $NINE_2$). One can wonder what kind of structure will result, if one proceeds from the truth values of a *constructive* logic.

Indeed, if we take seriously the *epistemical* justification of the ‘unusual’ combinations of truth values from [12] and the *informational* interpretation from [5] and [6], one can suggest that these combinations look most natural within a *constructive framework*. In fact, taking into account Heyting’s famous characterization of intuitionistic logic as a ‘logic of knowledge’ [27], Grzegorzczuk’s interpretation of his semantics for intuitionistic logic in terms of information states [24], Wansing’s consideration of Nelson’s system as a natural logic of information processing [50] and many similar interpretations given by other authors, one can expect that the ‘multivaluational approach’ should suit a constructive logic very well.

The aim of the present paper is to outline some essential features of the truth value space originating from the constructive truth values. We start from a detailed consideration of the basic truth values that we find in various constructive logics, and then we show how to generalize them using the idea of multivaluations.

2 Truth and falsity in constructive logic

What does it mean for a logic system to be constructive? There are a lot of systems that claim to be such, the most famous and well-studied among them being the intuitionistic logic **H** first completely formalized in [25]. But we also have various systems which are weaker than **H**, e.g. minimal logic by [28], as well as the class of the so-called *intermediate logics* which are stronger than **H** but weaker than classical logic. Furthermore, we have some important traditions defending versions of constructivism essentially different from the one asserted by intuitionism and related systems, e.g. the ‘constructive mathematical logic’ of [32] or the logic of ‘constructible falsity’ of [33]. Clearly, there should exist some distinctive features (*criteria*), so that any logical system turns out to be constructive by possessing these features, and non-constructive by lacking them. One such feature often mentioned in the literature is of a syntactic nature. It is the well-known *disjunction property*: if a formula $A \vee B$ is provable in the system, so is at least one of the formulas A or B . This property is very important for a characterization of a logic as one that represents a constructive pattern of reasoning. However, as our concern in the present paper is mainly the semantical one, we leave the disjunction property aside and dwell further on some semantic peculiarities which all the constructive logics have (or at least should have) in common.

Taken from a semantical perspective every constructive logic adopts what can be dubbed a ‘constructive understanding of truth’. Roughly, this understanding can be expressed as follows: *to be true means to be constructively (effectively) proven*. That is, a sentence is true if and only if there is a (constructive) proof of the sentence. This is indeed a very rough explication which needs further specifications.

First, to avoid circularity one has to explain the term ‘constructively’ in a way not involving this term. Usually this has been done by means of some kind of inductive procedure explaining what is a constructive proof of atomic sentences and then extending this explanation to the conjunctive, disjunctive, implicative and negative sentences, as well as to the universal and existential ones. For example, it is common in a constructive logic to interpret a constructive proof of an existential sentence $\exists x P(x)$ as such an effective procedure that gives us a certain object a and a proof that $P(a)$ holds.

Second, it appears that the expression ‘there is a proof’ can be interpreted in different ways. What does one mean, when one says that ‘there exists’ a proof of some sentence? Is it some kind of ‘potential existence’ (in some ‘possible world’), or does one claim a real ‘here and now’ existence? We find in the literature at least two different attitudes with respect to this problem.

One attitude, which presents the view of traditional intuitionism by Brouwer and Heyting, consists in demanding the *actual possession* of a proof for a sentence to be true. That is, a sentence is constructively true if and only if it is *actually proven* in a certain moment. In this way the nature of truth becomes essentially ‘time-bound’ [49, p. 248], or ‘tensed’ [8, p. 18].

But there is an alternative approach defended for example by Prawitz [32–34], which intends to interpret constructive truth as a ‘tenseless’ notion by speaking rather of *provability* than of actual proofs.⁴ Under such an interpretation truth means only the *possibility* of being proven, a possibility that need not be actualized. Dummett, whose work actually inspired the investigation of the notion of intuitionistic truth as a tenseless notion, remarked in [8, pp. 18–19], that both accounts could in principle be employed in rejecting the law of excluded middle. Therefore he seemed to be uncertain about the question of what kind of interpretation should be preferred as an intuitionistically acceptable understanding of existence of a proof. In his later writings, however [9, 10] he pointed out some essential perplexities connected with the tenseless understanding of truth. For example, he noticed that it forces us to accept an equation between the truth of the negation of some statement and the non-existence of its proof or verification [9, p. 285]. This circumstance has been remarked upon in [8], although there he did not characterize it as a serious difficulty. As he put it:

On the first interpretation of ‘ A is true’, as significantly tensed, i.e. as meaning ‘ A has been proved’, the statement ‘ A is false’, that is ‘ $\neg A$ is true’, is much stronger than ‘ A is not true’. But when ‘ A is true’ is interpreted as tenseless, i.e. as meaning ‘We can prove A ’, then ‘ A is not true’ and ‘ A is false’ can be equated ... [8, p. 19]

We believe however that there is a real difficulty here, and moreover that this difficulty is crucial. In fact, it shows that the second interpretation of intuitionistic truth implies a wrong account of intuitionistic *falsity*.

⁴Prawitz prefers more generally to speak of ‘verification’ instead of ‘proof’, and Rabinowicz also presents the difference between above mentioned two positions as a difference between identifying the truth either with ‘actual verification’ or with ‘verifiability’ [39, p. 191]. However, if we deal with mathematical propositions (as the traditional intuitionism does), these terms generally coincide: ‘to be verified’ means for a mathematical proposition simply ‘to be proven’.

Let us first clarify the terminology. It is not always clear what this or that author means by the expression ‘sentence A is intuitionistically false’. As we can see, Dummett interprets this expression simply as ‘ $\neg A$ is true’, i.e. as another way of expressing the truth of a negated sentence. However, if we conceive of falsity as a semantic notion belonging to the same level as the notion of truth, we should rather interpret it as a direct metalanguage negation of truth, that is as a ‘non-truth’. The whole issue is of course the matter of a language convention about the usage of basic terms. Let us mark the definition ‘ A is false’ =_{df} ‘ A is not true’ as introducing a *purely semantic notion of falsity*, and characterize the definition ‘ A is false’ =_{df} ‘ $\neg A$ is true’ as introducing a *notion of falsity involving syntactic terms*. One has to be very cautious by dealing with these definitions. It might seem that, e.g. Heyting did not distinguish between them, when he wrote: ‘“The proposition p is not true”, or “the proposition p is false” means “If we suppose the truth of p , we are led to a contradiction”’ [26, p. 18]. However, one has to take into account that Heyting’s view was essentially non-semantic (or pre-semantic), he did not aim at developing any semantic framework, and confined himself to semiformal explanations of the intuitive meaning of logical constants.

If we wish to construct a genuine semantic account of intuitionistic falsity, we cannot simultaneously take *both* of the above mentioned definitions. Moreover, a semantic construction should not have object-language entities among its primitives. Thus, it would be incorrect to *define* a semantic notion of falsity using an object-language negation. This semantical notion represents rather what Heyting called ‘factual negation’ [26, p. 19], as opposed to a ‘mathematical’ (object-language) negation. Therefore we believe that in intuitionistic logic — as opposed to classical logic — the expressions ‘ A is not true’ and ‘ $\neg A$ is true’ should *not* generally be equated. That is, intuitionistic semantic and syntactic notions of falsity *do not coincide*, and in intuitionism the semantic notion of falsity is *not* directly represented by an object-language connective of negation. The main difference is that ‘ $\neg A$ is true’ is and should be — according to intuitionistic paradigm — a *constructive statement*, whereas ‘ A is not true’ is not obligatorily constructive and indeed should not be. But a ‘tenseless’ interpretation of the notion of proof forces us to accept the unwelcome equation above. Therefore we reject this interpretation, and adopt an ‘orthodox’ (Brouwer and Heyting) understanding of constructive truth as ‘having an effective procedure for constructing a proof’.⁵

Consider now in more detail what one may call an ‘asymmetry between intuitionistic truth and intuitionistic falsity’. As already noted, this asymmetry finds its expression in the fact that intuitionistic truth, as opposed to intuitionistic falsity, is a constructive notion. Its important feature consists in the so-called ‘hereditary property’ (often also called *monotonicity*). A true statement remains always true: once proven, a sentence is proven for all times. This property surely is quite a strong idealization, but this idealization constitutes an essential feature of a constructive approach to scientific knowledge. Now, it is clear that a false sentence (understood as being not proven) should not necessarily possess this property, i.e. a false sentence need not remain false in the future. We may well have no proof of a sentence today, but tomorrow we can find this proof, and in this way the sentence ceases to be false and becomes true! However, the falsity of a sentence is preserved backwards: if a sentence is false now, it also means that it was false yesterday. Thus, in the course of our knowledge development the set of true sentences is always growing, while the set of false sentences is shrinking.

One might raise here the following interesting questions. How could it happen that a constructive logic (which the intuitionistic logic certainly is) adopts an essentially non-cons-

⁵There would be an even more conservative approach which would require having the actual proof in hand and not just a method for constructing it, but this seems not to reflect actual mathematical practice very well.

constructive notion of falsity? Does this fact negatively impact the very ‘constructivity’ of intuitionistic logic, and if it does not — why? Answering these questions, we have to remember that intuitionism as an innovative direction in the foundations of mathematics and logic was proclaimed and first developed at a ‘pre-semantic’ stage of the history modern logic. At those times (roughly the first 35 years of the 20th century) such words as ‘truth’ and ‘falsity’, which we use now as strict semantic categories, were often used ‘semiformally’ as a helpful instrument of explicating terms of the object language and explaining facts of the logical theories, with little concern about possible unintended consequences of such ‘informal’ usage. Even in [26] we do not find any semantic analysis in the modern (strict) meaning of the term. It is also worth mentioning that Brouwer preferred to speak not of ‘falsity’, but of ‘absurdity’, which was for him an approximate synonym of an object language negation. It seems that at the beginning stage of the development of intuitionism not many researchers even realized the non-constructivity of the intuitionistic notion of (semantic) falsity, just identifying this notion with intuitionistic negation.

Furthermore, we should take into account the importance of distinguishing between an object language and a metalanguage. The statements that we make in a metalanguage such as ‘sentence A is true’ (‘ A is proven’) or ‘sentence A is false’ (‘ A is not proven’) describe some facts *about* the state of our knowledge at a certain moment. These statements describe our theory, e.g. a mathematical theory, saying which sentences are in the theory and which are not, but *as such* they do not belong to the theory. It appears, however, that some facts *about* a theory directly correspond to certain sentences *in* this theory. All so-called ‘positive facts’ are of such character. If A is proven, then A belongs to our theory and vice versa. But, as opposed to this, the ‘negative descriptions’ often do not directly correspond to any sentence of our theory, they remain merely the statements ‘about matters of fact’ [26, p. 18]. A mere absence of a proof of A does not give us enough reason for including $\neg A$ in the theory. This slippage allows the object language of intuitionistic theories to remain constructive, even though the intuitionistic notion of falsity is not.

The intuitionistic conception of falsity is not the only one that we find in a constructive tradition. Indeed, one of the constructive logics was directly motivated by the idea of introducing a notion of falsity which is different from the one we meet in intuitionistic logic. This is the logic of ‘constructible falsity’ proposed in [33]. The main motivation for developing this logic was just to remove the above mentioned asymmetry between truth and falsity, and to conceive the later ‘in a fashion analogous to that for intuitionistic truth’ [1, p. 231].

Indeed, in an ‘orthodox’ intuitionistic theory falsity is a superfluous notion. If we understand it as a simple abbreviation for ‘non-truth’ (where ‘non’ is a metalanguage negation), it can be easily eliminated from the discourse, as is usually done in formulating semantics for intuitionistic logic. It appears that all necessary semantic notions can be introduced by means of the notion of truth and the usual logical connectives of a metalanguage. In contrast, Nelson’s idea was to consider falsity as a really autonomous notion introduced independently from the notion of truth and in a similar way. [33] presents a concrete construction of such kind for a particular formal system of number theory N . Nelson specifies the notion of truth by defining a predicate ‘the natural number a P-realizes the formula A ’. Then he introduces a ‘correlative concept of constructible falsity’ [33, p. 17] by defining a predicate ‘the natural number a N-realizes the formula A ’. Thomason explains this idea more generally as follows:

The falsity, for instance, of a formula such as $P(a)$ at a stage of construction is not defined in terms of the failure of the individual named by a to have the property corresponding to P ; rather, this falsity is conceived of as a feature which is discovered

directly, as the truth of $P(a)$ is discovered. [48, p. 251]

If we try to interpret such an understanding using a ‘proof-theoretic’ terminology, we may introduce side by side with the notion of proof a parallel notion of *disproof*, or *refutation*. Now, similarly to the interpretation of the expression ‘ A is (constructively) true’ as ‘ A is (constructively) proven’, we may interpret the expression ‘ A is (constructively) false’ as ‘ A is (constructively) disproved’, or ‘ A is (constructively) refuted’ (cf. [31]). In this way the information regarding the falsity of some sentence also represents a ‘constructive knowledge’ which is subject to the hereditary property: what is false at a given stage of our theory development, remains false at any later stage. The set of false sentences in this constructive sense — just as the set of true sentences — is growing.

Summarizing, examining the variety of constructive logics we find that they all share a similar — constructive — conception of truth, but some of them differ in their conception of falsity. We can basically single out two different conceptions of falsity in constructive logics. In one conception (intuitionistic logic) falsity is interpreted in a non-constructive fashion: ‘ A is false’ means simply ‘ A is not proven’. Another conception (Nelson logic) presents falsity as a constructive counterpart of the notion of truth: ‘ A is false’ means here ‘ A is disproved’, or ‘ A is refuted’.

3 Two ways of implementing the multivaluations into a constructive semantics

How can we extend the ‘multivaluational approach’ described in Section 1 to the realm of constructive truth values? Well, it depends on what kind of constructive logic we have in mind, more specifically, which pair of truth and falsity we wish to combine. We have at least two possibilities. We can consider either the pair

(constructive truth; intuitionistic falsity)

or the pair

(constructive truth; constructive falsity).

Let us start with Nelson’s logic of constructible falsity.⁶ [48] presents a Kripke-style semantics for this logic. The specific feature of this semantics is that it introduces both **T** and **F** as primitive notions, defining separately truth and falsity conditions for compound formulas. However, these truth values are still connected through the principle of Unique Valence, although the principle of Bivalence does not generally hold. Using the colourful terminology of K. Fine, we may say that Thomason’s semantics allows ‘gaps’, but forbids ‘gluts’. This fact shows that truth and falsity in Nelson’s logic still depend on each other, and this is a deep source for the paradoxes of implication and of the consequence relation. Indeed, the formulas $A \rightarrow (B \rightarrow A)$ and $\sim A \rightarrow (A \rightarrow B)$ are valid in the standard formulations of Nelson’s system **N**.

[14] investigates a collection of *generalized Nelson logics* which emerge when we consider possible combinations of various semantic conditions: conditions on the accessibility

⁶Although Nelson speaks of ‘constructible’ falsity, we will tend to use the term ‘constructive’ falsity in analogy to ‘constructive truth’ and for the reasons given in the last section, namely that we want to suggest having an actual disproof or at least the method for constructing one.

relation, truth and falsity definitions for implication, ‘gaps’ and/or ‘gluts’ in relation to constructive **T** and constructible **F**. In the present paper we sketch a general four-valued semantics (which is in effect a particular case of a multivaluational semantics) that determines a relevant first-degree entailment for formulas of Nelson logic. Let \mathcal{L}_N be the set of all the formulas formulated in the language of Nelson logic. Our task is to define a relation that determines all the valid relevant entailments between any two formulas from \mathcal{L}_N .

A *Nelson-frame* is a structure $\mathfrak{U}_N = \langle W, \sqsubseteq, \Vdash_T, \Vdash_F \rangle$, where W is a non-empty set, \sqsubseteq is a partial order on W , and \Vdash_T and \Vdash_F are two different forcing relations between elements of W and sentences from \mathcal{L}_N , defined for any atomic proposition p_i and subject to the following *hereditary conditions* (for any α and β from W):

$$\alpha \Vdash_T p_i \text{ and } \alpha \sqsubseteq \beta \Rightarrow \beta \Vdash_T p_i. \quad (\text{HC}_T)$$

$$\alpha \Vdash_F p_i \text{ and } \alpha \sqsubseteq \beta \Rightarrow \beta \Vdash_F p_i. \quad (\text{HC}_F)$$

Truth and falsity conditions for compound formulas are defined as follows:

$$\alpha \Vdash_T \sim A \Leftrightarrow \alpha \Vdash_F A. \quad (N \sim_T)$$

$$\alpha \Vdash_F \sim A \Leftrightarrow \alpha \Vdash_T A. \quad (N \sim_F)$$

$$\alpha \Vdash_T A \wedge B \Leftrightarrow \alpha \Vdash_T A \text{ and } \alpha \Vdash_T B. \quad (N \wedge_T)$$

$$\alpha \Vdash_F A \wedge B \Leftrightarrow \alpha \Vdash_F A \text{ or } \alpha \Vdash_F B. \quad (N \wedge_F)$$

$$\alpha \Vdash_T A \vee B \Leftrightarrow \alpha \Vdash_T A \text{ or } \alpha \Vdash_T B. \quad (N \vee_T)$$

$$\alpha \Vdash_F A \vee B \Leftrightarrow \alpha \Vdash_F A \text{ and } \alpha \Vdash_F B. \quad (N \vee_F)$$

$$\alpha \Vdash_T A \supset B \Leftrightarrow \forall \beta \sqsupseteq \alpha (\beta \Vdash_F A \text{ or } \beta \Vdash_T B). \quad (N \supset_T)$$

$$\alpha \Vdash_F A \supset B \Leftrightarrow \alpha \Vdash_T A \text{ and } \alpha \Vdash_F B. \quad (N \supset_F)$$

Because we are interested in the entailment relation, rather than ‘valid formulas,’ we do not introduce a notion of validity (as well as ‘validity in a frame’). Instead we first define that A *relevantly entails* B in a frame $\mathfrak{U} = \langle W, \sqsubseteq, \Vdash_T, \Vdash_F \rangle$ iff $\forall \alpha \in W (\alpha \Vdash_T A \Rightarrow \alpha \Vdash_T B)$. Let \mathfrak{F} be the class of all frames. Then a proposition A *relevantly entails* proposition B (in symbols $A \models B$) iff $\forall \mathfrak{U} \in \mathfrak{F} \forall \alpha \in W (\alpha \Vdash_T A \Rightarrow \alpha \Vdash_T B)$.

Consider an informal motivation of this semantics. W can be interpreted as a set of theoretic constructions, or a set of states of a constructive theory at different stages of its development. The relation \sqsubseteq is then a possible time-relation between theoretic constructions: $\alpha \sqsubseteq \beta$ means that β is a possible result of developing α . Another way of interpreting W and \sqsubseteq is to explicate it as a set of ‘pieces of information’ and \sqsubseteq as the ‘information order’. We would like to stress that in the case of a constructive logic any element from W represents *constructive information*, and \sqsubseteq thus reflects a growing of constructivity of our knowledge.

Two forcing relations \Vdash_T and \Vdash_F stand respectively for constructive truth and constructive falsity. The expression ' $\alpha \Vdash_T A$ ' can be read as 'the piece of information α *forces the (constructive) truth* of the sentence A ' and ' $\alpha \Vdash_F A$ ' as 'the piece of information α *forces the (constructive) falsity* of the sentence A '. Alternatively one can understand ' $\alpha \Vdash_T A$ ' as 'sentence A is constructively proven within a theoretical construction α '. The relation \Vdash_F , being a constructive counterpart of constructive truth, means not a mere absence of a proof, but a presence of some 'negative' construction that directly *refutes* A . That is ' $\alpha \Vdash_F A$ ' means that theoretical construction α refutes A . Then both hereditary conditions HC_T and HC_F express the constructive character of the forcing relations for truth and for falsity. The positive as well as the negative information should be growing in the course of our knowledge development. Note, that unlike in Thomason's semantics, we do *not* restrict the valuation of formulas to simply ' T , F , or undefined' [48, p. 252]. Now a valuation can also be 'over-defined', i.e. *both* $\alpha \Vdash_T A$ and $\alpha \Vdash_F A$ can take place for the same α and A . In this way we get a 'four-valued' semantics for Nelson's logic, and this semantics determines a relation of relevant entailment for formulas of this logic.

In [44, 46, 45] and [47] the multivaluational approach was applied to the truth values of intuitionistic logic in order to obtain a relation of relevant entailment for intuitionistic sentences. Thus, we deal with \mathcal{L}_H — the language of intuitionistic propositional logic. An *intuitionistic frame* is a structure $\mathfrak{U}_H = \langle W, \sqsubseteq, \Vdash_T, \Vdash_F \rangle$, where W , \sqsubseteq and \Vdash_T are as in Nelson-frame above, and \Vdash_F is another forcing relation for a non-constructive (intuitionistic) falsity subject to the following 'backward' hereditary condition:

$$\beta \Vdash_f p_i \text{ and } \alpha \sqsubseteq \beta \Rightarrow \alpha \Vdash_f p_i. \quad (HC_f)$$

The definitions of truth and falsity conditions for compound formulas:

$$\alpha \Vdash_T \sim A \Leftrightarrow \forall \beta \sqsupseteq \alpha (\beta \Vdash_f A). \quad (H \sim_T)$$

$$\alpha \Vdash_f \sim A \Leftrightarrow \exists \beta \sqsupseteq \alpha (\beta \Vdash_T A). \quad (H \sim_f)$$

$$\alpha \Vdash_T A \wedge B \Leftrightarrow \alpha \Vdash_T A \text{ and } \alpha \Vdash_T B. \quad (H \wedge_T)$$

$$\alpha \Vdash_f A \wedge B \Leftrightarrow \alpha \Vdash_f A \text{ or } \alpha \Vdash_f B. \quad (H \wedge_f)$$

$$\alpha \Vdash_T A \vee B \Leftrightarrow \alpha \Vdash_T A \text{ or } \alpha \Vdash_T B. \quad (H \vee_T)$$

$$\alpha \Vdash_f A \vee B \Leftrightarrow \alpha \Vdash_f A \text{ and } \alpha \Vdash_f B. \quad (H \vee_f)$$

$$\alpha \Vdash_T A \supset B \Leftrightarrow \forall \beta \sqsupseteq \alpha (\beta \Vdash_f A \text{ or } \beta \Vdash_T B). \quad (H \supset_T)$$

$$\alpha \Vdash_f A \supset B \Leftrightarrow \exists \beta \sqsupseteq \alpha (\beta \Vdash_T A \text{ and } \beta \Vdash_f B). \quad (H \supset_f)$$

The definition of the entailment relation is formally like that in Nelson models. In the case of intuitionistic models this definition determines the relation of relevant entailment for

intuitionistic sentences.⁷

Note, that the usual condition is $\alpha \Vdash_f A \Leftrightarrow \alpha \not\Vdash_T A$, and under this condition we get just a semantics for the intuitionistic logic **H**. However, in a multivaluational semantics we do not take this condition, and this allows situations when a sentence is both intuitionistically true and false, as well as situations when a sentence is neither true nor false intuitionistically.

The relation \Vdash_f is much weaker than \Vdash_F , and intuitively the expression ' $\alpha \Vdash_f A$ ' could be understood as 'the sentence A is *rejectable* within the theoretical construction α '. This is a 'rejectability relation', saying that *so far* we do not have enough reasons for including the sentence in our theory (or, we have some reasons for not including the sentence in a theory). Thus, we mark the sentence as rejectable (the sentence is *suspected*), although this does not exclude the possibility that we can later decide to include it in our theory. In this way \Vdash_f behaves just as the intuitionistic (non-constructive) falsity should: it is subject to HC_f , but is not subject to anything like HC_F .

Thus, combining the truth-values that we find in constructive logics, we have two different constructive 'four-valued semantics': one for a 'relevant Nelson logic' another for a 'relevant intuitionistic logic'.

4 Making the picture complete: the notion of a non-constructive truth

So far we have dealt only with two separate pairs of truth values occurring in the semantics for constructive logics. Our next task is to explore the possibility of a unified approach to these truth values and to combine all of them within *one* general framework. Let us start with a simple 'mechanical' gathering of \Vdash_T , \Vdash_F and \Vdash_f into *one set*. As far as we investigate the truth values 'in themselves' as 'abstract entities' (in a Fregean sense), we may omit any reference to their 'relational character'. That is, when we do not explicitly talk about some sentence having some value in some 'possible world', we just omit the sign \Vdash , and simply write **T** for the constructive truth, **F** for the constructive (constructible) falsity and **f** for the intuitionistic (non-constructive) falsity. Thus, we get the following set of truth values: $\{\mathbf{T}, \mathbf{F}, \mathbf{f}\}$. Now it becomes evident that the 'four-valued semantics' considered in Section 1 is merely a particular case of a more general method that we called a 'multivaluational approach'. Indeed, unifying *two* sets of initial truth values we get a new base consisting of more than two elements. Correspondingly, the resulting generalized truth value space will contain more than four elements.⁸

Considering the set $\{\mathbf{T}, \mathbf{F}, \mathbf{f}\}$, one can get a justified impression that it is somehow 'lopsided'. Moreover, we may notice that this 'lopsideness' is in fact twofold. First, this set contains only *one* entity for truth (**T**), but *two* entities for falsity (**F**, **f**). Second, it contains only *one* non-constructive entity (**f**), but *two* constructive entities (**T**, **F**). It is not difficult to see, why such an asymmetric situation occurs: the constructive falsity (unlike the constructive truth) is missing its non-constructive counterpart. Thus, to make the picture complete, we have to introduce among the basic truth-values the notion of a *non-constructive truth*.

We can understand such non-constructive truth as a specific epistemic attitude, when one

⁷See also the formulation of this semantics in terms of *intuitionistic states descriptions* in [43], where the condition HC_f is omitted. [16] sketches a similar generalization of Kripke models for intuitionistic logic (with no connection to the intuitionistic entailment) by introducing 'weak' Kripke models. These models make use of *signed formulas*, i.e. expressions of the forms TX and FX . Using this style of notation one would write $\alpha \Vdash TX$ and $\alpha \Vdash FX$ rather than $\alpha \Vdash_T A$ and $\alpha \Vdash_F A$.

⁸The idea to consider generalized truth values as subsets of a set containing more than two elements (**T** and **F**) has been also expressed in [29, p. 46].

considers a sentence *acceptable*, although it may well be that no proof of the sentence is so far present. This is a notion of truth as (maybe temporal) acceptability. If we (for some reason) need a sentence, we *might* accept it, although later we can change our mind and decide that the sentence is not acceptable any more (e.g. when we obtain a refutation of this sentence). This motivates introduction of a new truth-value that can be labelled as *t*.

The expression ' $\alpha \Vdash_t A$ ' can be interpreted as '*A* is acceptable within α ' (or maybe '*A* is *tolerated* within α '). Note that if we would like to incorporate the non-constructive truth into a Kripke frame, we should take the following condition analogues to HC_f :

$$\beta \Vdash_t p_i \text{ and } \alpha \sqsubseteq \beta \Rightarrow \alpha \Vdash_t p_i. \quad (\text{HC}_t)$$

Non-constructive truth, being a *non-constructive* truth value, should be preserved backward!

As in case with the relationships between constructive truth and non-constructive falsity, we do not generally take the usual ('normal') condition $\alpha \Vdash_t A \Leftrightarrow \alpha \not\Vdash_F A$. A multivaluational approach implies 'independent' treatment of all four truth values. Thus, 'abnormal' situations can occur, when a sentence is refuted, but still marked as acceptable, as well as not acceptable although proven. Such situations may well happen when we have an overflow of information, inconsistent data, or when an agent behaves not in a rational way. This of course does not prevent us from introducing some possible dependencies between the truth values on a later stage of analysis (see Section 6).

There is one important difference between constructive and non-constructive truth values. The constructive values represent *actually accomplished constructions* (proofs and disproofs). As such they carry *constructive information*, or implement *constructive knowledge*. As opposed to this, non-constructive truth values as such do not mean presence of any completed construction.

5 Generalized truth-value space of constructive logic. The trilattice

Thus, as the basis of the generalized constructive truth-value space we consider the following set of initial truth values: $\mathcal{I} = \langle \mathbf{T}, \mathbf{F}, \mathbf{t}, \mathbf{f} \rangle$. Let us summarize here the informal meaning of the elements of this set. We have:

- \mathbf{T} — a sentence is constructively *proven*;
- \mathbf{F} — a sentence is constructively *refuted*;
- \mathbf{t} — a sentence is *acceptable*;
- \mathbf{f} — a sentence is *rejectable*.

Applying the idea of multivaluation to this set, we have to consider *all the possible combinations* of these elements, i.e. the power set of \mathcal{I} . It is easy to see that this gives us 16 generalized truth-values, or 'multivalues':

$$\begin{aligned} \mathcal{P}(\mathcal{I}) = & [\{\}, \{\mathbf{T}\}, \{\mathbf{F}\}, \{\mathbf{t}\}, \{\mathbf{f}\}, \{\mathbf{T}, \mathbf{F}\}, \{\mathbf{T}, \mathbf{t}\}, \{\mathbf{T}, \mathbf{f}\}, \{\mathbf{F}, \mathbf{t}\}, \{\mathbf{F}, \mathbf{f}\}, \{\mathbf{t}, \mathbf{f}\}, \\ & \{\mathbf{T}, \mathbf{F}, \mathbf{t}\}, \{\mathbf{T}, \mathbf{F}, \mathbf{f}\}, \{\mathbf{T}, \mathbf{t}, \mathbf{f}\}, \{\mathbf{F}, \mathbf{t}, \mathbf{f}\}, \{\mathbf{T}, \mathbf{F}, \mathbf{t}, \mathbf{f}\}]. \end{aligned}$$

In what follows, we write \mathbf{N} for the empty multivalue, and \mathbf{A} for the multivalue representing the set that includes *all* the basic truth values. Also we will omit braces and commas in writing down the other generalized truth-values.

Note again that we most abstractly start from a 'pure combinatorial' point that does not take into account any philosophical or other considerations about possible interrelations between the basic truth values. Such an approach results in the most general truth value space

originating from the given basis. However, after a while we will consider such possible inter-relations, and consider conditions that would restrict the general space of 16 values to some smaller spaces. We will return to this point later on, and will show how the structures of these smaller spaces appear to be substructures of the general truth value space.

Let us consider the notion of a bilattice once again. From a philosophical point of view any bilattice can be thought of as a structure that represents degrees of *two* basic properties which are in a sense *essential* for the elements that form the bilattice. The most fundamental property is the one of *having information*. In fact, the nature of the elements themselves plays a minor role when we think of ‘pure information’. The amount of the elements is the important thing: the more elements a set consists of, the more information it bears. The second property is the one of *being true*. This property is essential, because we deal here with a *truth-value* space. It is important that every element of a bilattice has each of these two properties in a *certain degree*. And the two partial orderings — \leq_i and \leq_t — ‘organize’ elements in accordance with the possessed degree of information and truth respectively.

Now one can notice that as soon as we turn to a *constructive* truth value space, we have to take into account another important property that essentially characterizes each element of this space. This is the property of *constructivity*. Indeed, every multivaluation, combining constructive and non-constructive truth values, ascribes to every resulting generalized value not only some degree of information and truth, but also a certain degree of constructivity. This means that we obtain another partial ordering — \leq_c — which represents an increase in constructivity (or a decrease in non-constructivity) among the elements of $\mathcal{P}(\mathcal{I})$. Under this new partial ordering we also have a complete lattice, and thus, we can introduce the notion of a *trilattice* as follows:

DEFINITION 5.1

A *trilattice* is a structure $\mathcal{T} = (S, \leq_i, \leq_t, \leq_c)$ such that S is a non-empty set and (S, \leq_i) , (S, \leq_t) , (S, \leq_c) are complete lattices.

If S is $\mathcal{P}(\mathcal{I})$, then **A** and **N** are respectively the lattice top and bottom relative to \leq_i , **Tt** and **Ff** the bounds relative to \leq_t , and **TF** and **tf** relative to \leq_c . Indeed, **A** and **N** are the most and the least *informative* elements of $\mathcal{P}(\mathcal{I})$, **Tt** and **Ff** are the most and the least *true* of its elements, and **TF** and **tf** are the most and the least *constructive* elements.

Let us be more explicit by introducing the three ordering relations on $\mathcal{P}(\mathcal{I})$. For every x in $\mathcal{P}(\mathcal{I})$ mark by x^{Tt} the part of x which contains exactly those of the values **T** or **t** that are in x . The sets x^{Ff} , x^{TF} , x^{tf} can be defined analogously. Then we set up the following:

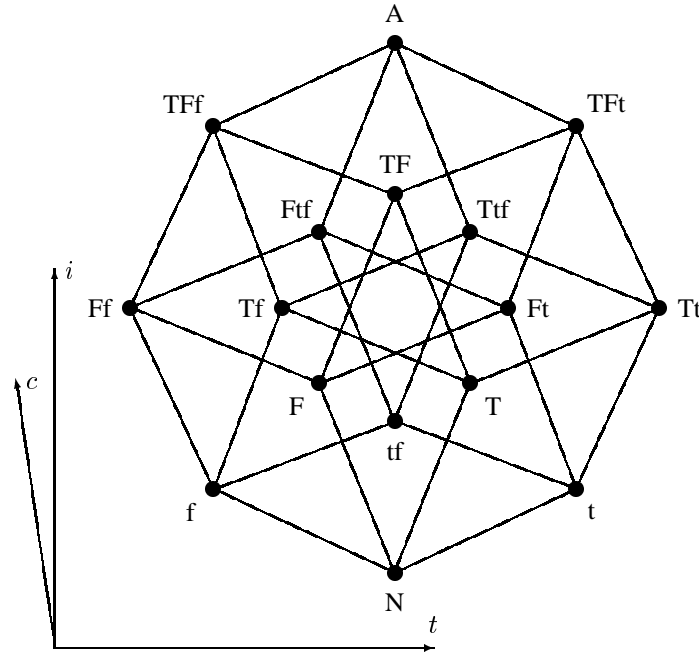
DEFINITION 5.2

For every x, y in $\mathcal{P}(\mathcal{I})$

1. $x \leq_i y \Leftrightarrow x \subseteq y$;
2. $x \leq_t y \Leftrightarrow x^{Tt} \subseteq y^{Tt}$ and $y^{Ff} \subseteq x^{Ff}$;
3. $x \leq_c y \Leftrightarrow x^{TF} \subseteq y^{TF}$ and $y^{tf} \subseteq x^{tf}$.

In this way we obtain the most general trilattice generated by \mathcal{I} — the trilattice *SIXTEEN*₃ that can be presented by a triple Hasse diagram as in Figure 5. Here, in comparison with diagrams for bilattices, a new *c*-axis represents the third ‘dimension’ of the trilattice, namely, the constructive one.⁹

⁹Lakshmanan and Sadri were apparently the first who considered (although in a quite different context) the notion of a trilattice in [30] — a very interesting article which unfortunately we were not aware of when working

FIGURE 5. Trilattice $SIXTEEN_3$ (projection $i - t$)

Clearly meets and joins exist for all three partial orderings. We will use \wedge and \vee for meet and join under \leq_t , \sqcap and \sqcup for those that correspond to \leq_i , \triangle and ∇ for the lattice operations under \leq_c .

Intuitively \wedge and \vee are just logical conjunction and disjunction. It is interesting to compute in each case the outputs of applying these connectives to generalized truth values. While $\mathbf{T} \wedge \mathbf{F}$, $\mathbf{T} \vee \mathbf{F}$, $\mathbf{t} \wedge \mathbf{f}$ and $\mathbf{t} \vee \mathbf{f}$ present no surprise, the behaviour of $\mathbf{T} \wedge \mathbf{t}$ may seem rather unexpected. As we can see, in the latter case the result is \mathbf{N} . That is, the conjunction of two ‘truths’ gives ‘nothing’. However, if we consider this situation in more detail, it appears that it has a quite natural intuitive justification, and the result is just as it should be. Indeed, a conjunction is true iff both conjuncts are true. This should hold for both \mathbf{T} and \mathbf{t} .

on our paper. We are thankful to the anonymous referee for pointing us to this article. In [30, p. 257], one can find a similar motivation of extending Ginsberg’s bilattices by a third ordering. However [30] deals with an algebra defined on a set of interval pairs rather than on a set of truth values. It aims at constructing a *probabilistic* calculus which is used for developing a suitable framework for *probabilistic deductive databases*. Trilattices as conceived in [30] should provide an ‘algebraic footing’ for this particular framework. The interval pairs present just probability ranges of beliefs and doubts (called ‘confidence levels’). The third ordering relation in the trilattice of confidence levels is a *precision* ordering, understanding information to be more or less precise (in fact, more or less probable). Clearly the meanings of the precision ordering in [30] and the constructivity ordering in the present paper are quite different. Moreover, we believe that our approach provides a more general and uniform framework for treating logical trilattices. We do not restrict our consideration to a particular (e.g. probabilistic) account on the knowledge base systems, dealing most abstractly just with lattices defined on a set of truth values (as is traditional, e.g. in works by Belnap, Ginsberg, Fitting and others).

cannot have the value \mathbf{T} (even as a constituent part), because it is not the case that *both* of the conjuncts have this value, and for exactly the same reason it cannot contain \mathbf{t} . But it would be also incorrect to ascribe to a conjunction of two truths (with no ‘admixture’ of falsity at all) any of the values \mathbf{F} or \mathbf{f} ! Thus, the only remaining possibility is \mathbf{N} , as we get it under the logical ordering in $SIXTEEN_3$. The analogous observations hold for $\mathbf{F} \vee \mathbf{f}$ and the other similar cases.

The operations under the information ordering can be thought of just as the intersection and union of ‘pieces of information’. In [18] \sqcap has been interpreted as a ‘consensus’ operator, and likewise \sqcup considered to be a ‘gullibility’ operator.

As to \triangle and ∇ , they also allow quite a natural interpretation. Every element from a generalized constructive truth value space has both a constructive and non-constructive part. \triangle just intersects the constructive parts of two generalized truth values and unifies their non-constructive parts, whereas ∇ works the other way around. Loosely $x \triangle y$ is the ‘nearest’ element that has at least as much constructivity (‘constructive content’) as *both* x and y have, and at least as much non-constructivity as either of x or y has. And ∇ , vice versa, outputs a truth value which has at least as much non-constructivity as both x and y have, and at least as much constructivity as has either of these elements. Actually \leq_c is an algebraic representation of the accessibility relation in Kripke frames for intuitionistic and Nelson logics. Indeed, under \leq_c the constructive constituent of any truth value is preserved forwards (cf. HC_T and HC_F), whereas the non-constructive constituent has the property of a ‘backward preservation’ (cf. HC_t and HC_f). Incidentally, this observation shows that it is not quite correct (as is sometime done) to interpret the accessibility relation in Kripke (or Grzegorzczuk) semantics for intuitionistic logic just as a relation between pieces of information. In fact, it is rather a relation between *constructive parts* of pieces of information.

We wish to stress once again the importance of the multivaluational approach in producing generalized truth value spaces. For every such space it is important to single out its *base* — the set of initial truth values, subsets of which form the range of the multivaluational function. One can call a truth value space (monolattice, bilattice, trilattice, etc.) *perfect* if and only if the set of generalized truth values constituting this space is just the powerset of its base. Sometimes it is useful (to be able to distinguish between structures having the same number of different elements) to explicitly mark the base of a perfect generalized truth value space by its name. This can be done by introducing (when necessary) the appropriate superscripts, e.g. $FOUR_2^{T,F}$, $FOUR_2^{t,f}$, etc. If there is no threat of confusion, the superscripts may be omitted.

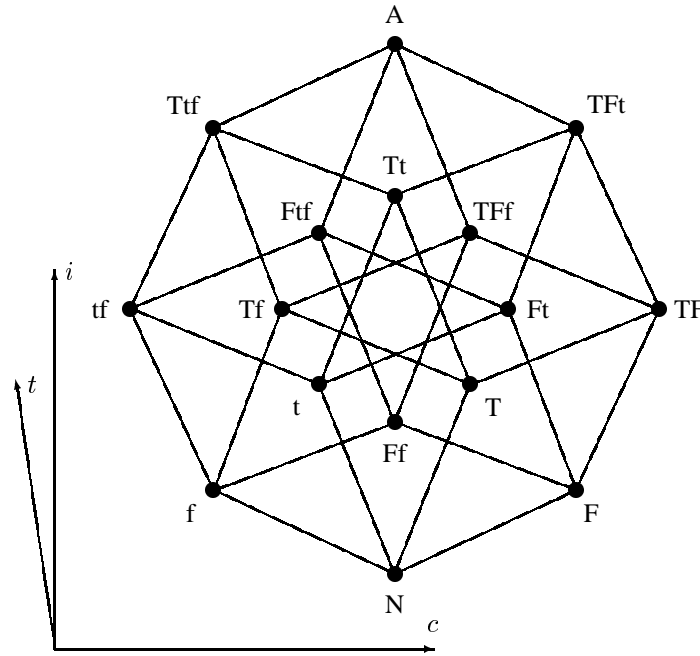
Let us provide more explanation of the diagram in Figure 5. One should imagine a ‘three-dimensional’ structure of this trilattice. Then it becomes clear that values \mathbf{T} , \mathbf{F} , \mathbf{t} and \mathbf{f} belong to the same *informational level*. Generally, $SIXTEEN_3$ has five informational levels, five logical levels and five levels of constructivity. These levels can be summarized as shown in Table 1.

We thus should warn the reader that axis c in Figure 5 is drawn very approximately to give only a rough idea about the third ‘dimension’ of $SIXTEEN_3$ (had it been drawn more precisely the axis c would have been visually coincident with the axis i). To avoid any confusion we present another projection of $SIXTEEN_3$ in Figure 6 (namely the projection on the plain $i - c$) displaying thus the ordering \leq_c in a precise way.

Note that Definition 5.1 is in sympathy with Ginsberg’s original definition of a bilattice from [21]. According to that definition a bilattice is just a set with two partial orderings each forming a lattice on this set, i.e. generating its own meet and join operators. [18] calls such

TABLE 1. Levels of $SIXTEEN_3$

	Information	Truth	Constructivity
1	N	Ff	tf
2	T, F, t, f	F, f, TFf, Ftf	t, f, Ttf, Ftf,
3	TF, Tt, Tf, Ft, Ff, tf	A, TF, Tf, Ft, tf, N	A, Tt, Tf, Ft, Ff, N
4	TFt, TFf, Ttf, Ftf	T, t, TFt, Ttf	T, F, TFt, TFf
5	A	Tt	TF

FIGURE 6. Trilattice $SIXTEEN_3$ (projection $i - c$)

structure a pre-bilattice. [22] presents a different definition of a bilattice, to the effect that a *negation* operator becomes a necessary ingredient of the whole structure. According to this latter definition every bilattice should have a negation operation that inverts the truth order while leaving the information (knowledge) order unchanged. This understanding is adopted by Fitting, Arieli and Avron, and some other authors. By defining the trilattice we return to the ‘spirit’ of the original definition from [21], because we believe that such definition better corresponds to a general understanding of a lattice just as a special type of partially ordered set. In this way we get a rather uniform picture that allows further generalizations: we single out *one* partial ordering for a lattice (a *monolattice*), *two* partial orderings for a *bilattice*, *three* partial orderings for a *trilattice* etc. The negation-like operations can be introduced at later

stages of consideration. As we shall see, specific properties of these operations may vary as well as the number of the operations themselves.

Having a trilattice, it is natural to consider at least three unary operations, each of which inverts one of the trilattice orderings and preserves the two others. The operation that inverts the logical ordering is in fact a logical negation. Fitting in [18] introduces on the bilattices the analogous operation with respect to the information ordering which he calls *conflation*. It is also possible to consider another unary operation that inverts \leq_c , but leaves \leq_t and \leq_i unchanged. Let us introduce some systematic names for these unary operations. We will generally call this type of operation an *inversion*, appropriately labelling the order which is inverted by a particular operation.

DEFINITION 5.3

Let \mathcal{T} be a trilattice. Then we can introduce the following unary operations on \mathcal{T} with the following properties.

1. *T-inversion* (\sim_t):
 - (a) $a \leq_t b \Rightarrow \sim_t b \leq_t \sim_t a$;
 - (b) $a \leq_i b \Rightarrow \sim_t a \leq_i \sim_t b$;
 - (c) $a \leq_c b \Rightarrow \sim_t a \leq_c \sim_t b$;
 - (d) $\sim_t \sim_t a = a$.
2. *I-inversion* (\sim_i):
 - (a) $a \leq_i b \Rightarrow \sim_i b \leq_i \sim_i a$;
 - (b) $a \leq_t b \Rightarrow \sim_i a \leq_t \sim_i b$;
 - (c) $a \leq_c b \Rightarrow \sim_i a \leq_c \sim_i b$;
 - (d) $\sim_i \sim_i a = a$.
3. *C-inversion* (\sim_c):
 - (a) $a \leq_c b \Rightarrow \sim_c b \leq_c \sim_c a$;
 - (b) $a \leq_i b \Rightarrow \sim_c a \leq_i \sim_c b$;
 - (c) $a \leq_t b \Rightarrow \sim_c a \leq_t \sim_c b$;
 - (d) $\sim_c \sim_c a = a$.

If the corresponding operation is defined on a given trilattice, we may call it a trilattice with *t-inversion*, *i-inversion* or *c-inversion*.

Consider *SIXTEEN*₃. It seems quite natural to define *t-inversion* (i.e. logical negation) on this trilattice so that it turns **T** into **F** and back, as well as **t** into **f** and vice versa. This produces a real truth table for negation (see Table 2).

TABLE 2. *T-inversion* (Nelson negation)

a	$\sim_t a$	a	$\sim_t a$	a	$\sim_t a$	a	$\sim_t a$
T	F	Tt	Ff	TFt	TFf	TF	TF
F	T	Ft	Ft	TFf	TFt	tf	tf
t	f	Ft	Tf	Ttf	Ftf	N	N
f	t	Ff	Tt	Ftf	Ttf	A	A

The last two columns of the table present the fixed points of *SIXTEEN*₃ relative to \sim_t so defined. The *t-inversion* defined by this table is in fact negation of Nelson's logic of constructible falsity.

‘Pure’ c -inversion should basically interchange **T** with **t** as well as **F** with **f**. The resulting definition is shown by Table 3. Again, the last two columns show the fixed points.

TABLE 3. C -inversion

a	$\sim_c a$	a	$\sim_c a$	a	$\sim_c a$	a	$\sim_c a$
T	t	Tf	Ft	TFt	Ttf	Tt	Tt
F	f	TF	tf	TFf	Ftf	Ff	Ff
t	T	tf	TF	Ttf	TFt	N	N
f	F	Ft	Tf	Ftf	TFf	A	A

As to pure i -inversion, it treats all the ‘*both*s’ truth values as fixed points while interchanging **N** with **A**, **T** with **Ttf**, **F** with **Ftf**, **t** with **TFt**, and **f** with **TFf** (see Table 4).

TABLE 4. I -inversion

a	$\sim_i a$	a	$\sim_i a$	a	$\sim_i a$	a	$\sim_i a$
T	Ttf	t	TFt	A	N	TF	TF
F	Ftf	f	TFf	N	A	Tt	Tt
Ttf	T	TFt	t	Tf	Tf	Ff	Ff
Ftf	F	TFf	f	Ft	Ft	tf	tf

A trilattice allows also another kind of inversion, namely an operation which simultaneously reverses *two* partial orderings with no impact on the third one.

DEFINITION 5.4

Let \mathcal{T} be a trilattice. Then we can introduce the following unary operations on \mathcal{T} with the following properties.

1. Tc -inversion (\sim_{tc}):
 - (a) $a \leq_t b \Rightarrow \sim_{tc} b \leq_t \sim_{tc} a$;
 - (b) $a \leq_c b \Rightarrow \sim_{tc} b \leq_c \sim_{tc} a$;
 - (c) $a \leq_i b \Rightarrow \sim_{tc} a \leq_i \sim_{tc} b$;
 - (d) $\sim_{tc} \sim_{tc} a = a$.
2. It -inversion (\sim_{it}):
 - (a) $a \leq_i b \Rightarrow \sim_{it} b \leq_i \sim_{it} a$;
 - (b) $a \leq_t b \Rightarrow \sim_{it} b \leq_t \sim_{it} a$;
 - (c) $a \leq_c b \Rightarrow \sim_{it} a \leq_c \sim_{it} b$;
 - (d) $\sim_{it} \sim_{it} a = a$.
3. Ci -inversion (\sim_{ci}):
 - (a) $a \leq_c b \Rightarrow \sim_{ci} b \leq_c \sim_{ci} a$;
 - (b) $a \leq_i b \Rightarrow \sim_{ci} b \leq_i \sim_{ci} a$;
 - (c) $a \leq_t b \Rightarrow \sim_{ci} a \leq_t \sim_{ci} b$;
 - (d) $\sim_{ci} \sim_{ci} a = a$.

Most interesting is the operation of tc -inversion. This operation not only establishes a connection between constructive and non-constructive truth values, but inverts also truth into falsity and back, having thus certain features of logical negation. We get a definition of tc -inversion (Table 5) if we interchange **T** with **f**, and **F** with **t**.

TABLE 5. Tc -inversion (De Morgan negation)

a	$\sim_{tc} a$	a	$\sim_{tc} a$	a	$\sim_{tc} a$	a	$\sim_{tc} a$
T	f	Tt	Ff	TFt	Ftf	Tf	Tf
F	t	TF	tf	TFf	Ttf	Ft	Ft
t	F	tf	TF	Ttf	TFf	N	N
f	T	Ff	Tt	Ftf	TFt	A	A

Table 5 shows that tc -inversion so defined is in fact *De Morgan negation*.

The last unary operation (\sim_{tc}) that can be introduced on a trilattice inverts simultaneously all three partial orderings. It is defined on $SIXTEEN_3$ by creating the set-theoretical complement (relative to the base of $SIXTEEN_3$) to every generalized truth-value. This operation has no fixed points.

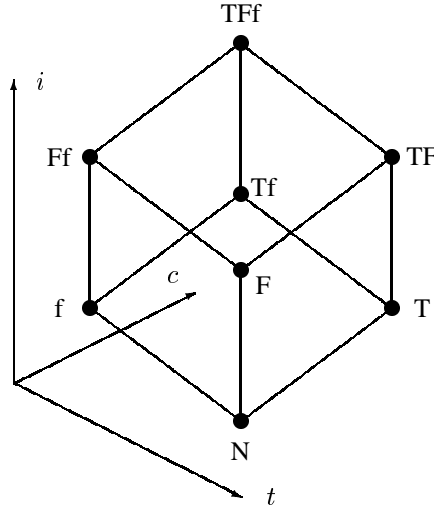
Since the detailed investigation of the properties of the trilattice operations goes far beyond the scope of the present paper, we leave it for future work.

6 Some interesting sublattices

Note that Definition 5.1 explicitly treats trilattices as a special kind of a *partially ordered set*. As is well known, this notion of *lattice as poset* is generally equivalent to the notion of *lattice as algebra*. In this latter sense $SIXTEEN_3$ can well be considered just a structure $\mathcal{T}^a = (\mathcal{P}(\mathcal{I}), \wedge, \vee, \sqcap, \sqcup, \Delta, \nabla)$. In this section we distinguish certain substructures on this general structure, and in this way consider some particular subspaces of the generalized truth value space represented by $SIXTEEN_3$. We also point out some important features of these subspaces.

First, we notice that $SIXTEEN_3$ contains all possible $FOUR_2$ -bilattices as sublattices. For example, we can consider a family of bilattices generated by the following bases: $\langle \mathbf{T}, \mathbf{F} \rangle$, $\langle \mathbf{T}, \mathbf{f} \rangle$, $\langle \mathbf{t}, \mathbf{F} \rangle$, $\langle \mathbf{t}, \mathbf{f} \rangle$. Note that we are still able to recognize in these bilattices all *three* partial orderings: \leq_i , \leq_t and \leq_c . Nevertheless it appears that two of the three ordering relations in each of these lattices either *coincide* or are *duals* of each other. That is, one of the three ordering relations is in a sense ‘degenerate’. Therefore the algebraic structure of these lattices remains essentially a bilattice (and *not* a trilattice). It is worth observing that the lattices $FOUR_2^{T,F}$ and $FOUR_2^{t,f}$ are *dual* in the sense that the constructive order of the former just coincides with its information order, whereas in the latter the constructive order is the reverse of the information order. $FOUR_2^{T,f}$ and $FOUR_2^{t,F}$ form dual structures in the same sense (with respect to the interrelations between their constructive orders and their logical orders).

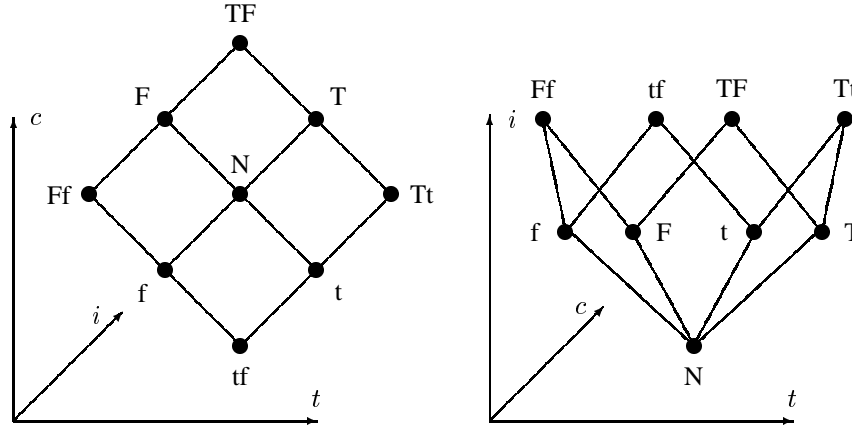
We can also consider an interesting ‘logic of falsity’ generated on the base $\langle \mathbf{F}, \mathbf{f} \rangle$ (the logic order is the dual of the information order) and the dual ‘logic of (mere) truth’ with the base $\langle \mathbf{T}, \mathbf{t} \rangle$ (the logic order coincides with the information order). It seems that the resulting bilattices may provoke deep philosophical interpretations.

FIGURE 7. Trilattice $EIGHT_3$

Another remarkable substructure of $SIXTEEN_3$ is the generalized truth value space originated on the ‘lopsided’ base from Section 4: $\langle T, F, f \rangle$ (union of the truth values from the Nelson and intuitionistic logics). It appears that the resulting generalized truth values also form a trilattice (namely, $EIGHT_3$ that is presented in Figure 7). The structure of $EIGHT_3$ displays the same three ordering relations, moreover here they are all explicitly distinct: \leq_i goes from N to Tff , \leq_t goes from Ff to T , and \leq_c is directed from f to TF .

Let us take $EIGHT_3$ as an example, and consider one interesting question (having in mind its general relevance to all kinds of lattices). $EIGHT_3$ apparently has more ‘potential’ ordering relations that those we singled out. For example, one could look at the direction from F to Tf (or vice versa). Should we state then, that $EIGHT$ is a *tetralattice* rather than a trilattice ($EIGHT_4$ rather than $EIGHT_3$)? Our answer to this question is ‘not at all’. A careful consideration shows that the ordering between F (as a top) and Tf (as a bottom) is not really new! Namely, this ordering represents an increase in *constructive falsity*. Tf is the least constructive *and* false element of $EIGHT_3$, and F is the most constructive *and* false element. Indeed, the notion of constructive falsity is clearly a combination of two ‘old’ notions, each of which is already represented in $EIGHT_3$ by a certain ordering relation.¹⁰

¹⁰‘Constructible falsity’ is, in a sense, analogous to the famous predicates ‘grue’ and ‘bleen’ considered by Goodman [23, pp. 74–81]. Generally, it is not difficult to define new predicates by combining the already existing ones, or by singling out some non-essential features of the objects. Thus, we do not claim that the properties of informativity, truth and constructivity are *the only* possible properties of the elements in the trilattice, and that \leq_i , \leq_t and \leq_c are *the only* possible orderings relations on it. Rather we state that these properties (and relations) are in a sense *essential* for the given structure. Or, more abstractly, that we just regard them as necessary for our analysis. Speaking rather informally one can require that the ordering relations which one singles out on some set should be ‘mutually independent’. The idea of ‘mutual independence’ can be explained as follows. Consider a structure with some partial orderings on it. Suppose we introduce an additional ordering relation. Then, first, we may wish that any such relation be *really new*, and not reducible to already existing relations (definable through them). But this is not enough. As already noted, every ordering relation represents a certain background property so that it orders the elements of a lattice with respect to the possessed degree of this property. Thus, and this is second, we may wish

FIGURE 8. Bi-and-a-half-lattice $NINE_{2.5}$

Note that the sublattices of $SIXTEEN_3$ considered so far were all perfect with respect to their bases. However, some additional conditions imposed on a generalized truth value space can produce structures that are not perfect. Such conditions set some dependencies between initial truth values, effectively forbidding some of their combinations. Let S be a subset of $\mathcal{P}(\mathcal{I})$. Consider the maximal truth value space satisfying the following conditions:

$$\forall a \in S (\mathbf{T} \in a \Leftrightarrow \mathbf{f} \notin a). \quad (T \times f)$$

$$\forall a \in S (\mathbf{F} \in a \Leftrightarrow \mathbf{t} \notin a). \quad (F \times t)$$

These conditions express an idea of ‘self-consistency’ (or rationality). The background intuition is as follows. We may well have a (syntactic) contradiction in our theory (\mathbf{TF}). However, even then we may wish to remain ‘consistent’ (i.e. *rational*) in our ‘epistemic behaviour’. If a sentence is proven, we should not reject this sentence as long as we preserve it in our theory. And dually in case of refutation — we should not accept (tolerate) refuted sentences. From a certain standpoint these conditions seem to be quite plausible. Consequently, they restrict our truth value space to nine truth values: $\{\mathbf{N}, \mathbf{T}, \mathbf{F}, \mathbf{t}, \mathbf{f}, \mathbf{TF}, \mathbf{Tt}, \mathbf{Ff}, \mathbf{tf}\}$. It may sound rather strange, but the structure we get from these truth values is in effect a bi-and-a-half-lattice! This lattice $NINE_{2.5}$ is presented in Figure 8. Left and right diagrams of the figure demonstrate different projections of $NINE_{2.5}$. One can clearly observe the complete lattices under \leq_t and \leq_c , but the information ordering fails to form a lattice. Under \leq_i we have merely a *semilattice* with \mathbf{N} as bottom, but with no top.

that any such background property be really a new property, i.e. that it is not definable (expressible) in terms of other background properties. If both conditions hold, we may say that the corresponding ordering relation is independent from the other relations, and altogether they are mutually independent. It would be an interesting task to define ‘mutual independence’ in a strict manner and to introduce it as an ‘official’ condition for the ordering relations in trilattices (and bilattices). Indeed, even in bilattices it is sometimes possible to find more ordering relations than the initial two. For example, why not to consider in $SEVEN_2$ (Figure 4) the ordering from \mathbf{df} to \mathbf{T} ? Nevertheless, this ordering also is of a ‘combinatorial’ character, therefore one can disregard it.

We finish this section by mentioning another possible condition, according to which constructive truth values, being stronger, ‘swallow up’ their non-constructive counterparts. This excludes from the resulting truth value space all the elements containing combinations **Tt** and **Ff**. This space appears to be another version of $NINE_{2.5}$.

7 Mathematical and philosophical observations. Possible generalizations

Figure 5 presents what in the theory of regular polytopes is known as a *hypercube*, more specifically, a *tesseract* [7, p. 123]. Now, how does a very-well known fact that a tesseract is a *four*-dimensional geometrical figure fit our characterization of $SIXTEEN_3$ as a *trilattice*, and its presentation using a *three*-dimensional coordinate space formed by *i*-, *t*-, and *c*-axes? Maybe we should revise our earlier account and recognize that $SIXTEEN_3$ is in fact a tetralattice, whereas a paradigmatic trilattice is presented by a ‘cube’ $EIGHT_3$?

In what follows we argue that too literal ‘introduction of ‘geometrical’ intuitions into lattice theory is misleading. We will show that it is not the amount of geometrical dimensions that determine the nature of a particular lattice, and that *both* $EIGHT_3$ and $SIXTEEN_3$ are trilattices. Although it is possible to generalize this notion and to speak of tetralattices, pentatlattices, etc., more generally, of *n*-lattices, it should be done along some essential lines of lattice theory and not of geometry. Nevertheless, it appears that the ‘geometrical’ notion of a *n*-space also finds its natural analogue in generalized truth value spaces.

Consider Table 6 which gives the number of various ‘geometrical entities’ in a sequence of structures (cf. [20, p. 45]). The first two columns of this table play the key role by

TABLE 6. ‘Geometrical’ elements of various structures

<i>n</i> -Space	Points	Lines	Squares	Cubes	Tesseracts
0	1	0	0	0	0
1	2	1	0	0	0
2	4	4	1	0	0
3	8	12	6	1	0
4	16	32	24	8	1

interpreting truth value spaces as lattices. The column ‘*n*-Space’ can be treated as presenting the number of initial truth values in the base of a possible perfect truth value space. The column ‘Points’ can be interpreted then as presenting the number of generalized truth values obtained by application of a multivaluational function to the elements of the corresponding base.

Now we can tell a general story of how the truth values are brought to existence and how various logics become possible. The first row of the Table 6 represents what can be called a ‘Hegelean world’ (see his *Science of Logic*, Book One: ‘The Doctrine of Being’). It starts with a ‘pure being’ which, according to Hegel, is in fact ‘nothing’ (‘complete emptiness’): the base of a truth value space in the first row is empty, it contains no elements. However, it would be incorrect to treat this space as a ‘non-being’, because it yet contains *one* element, originating from application of a multivaluational function to the empty set base.¹¹ This

¹¹Cf. the theory of Parmenides that not-being does not exist, and whatever exists is being.

only element — \emptyset — just embodies the ‘pure being’ in Hegelean sense. It is interesting to observe that contrary to Hegel’s thought, we have here a merely pre-logical stage of the world’s development, because no logic is possible with the only one truth value, whatever that truth value is.

It is the second row of Table 6 where logic comes to existence, and it appears that this logic is *classical logic*. Incidentally, the second row makes it clear how the classical truth value function can be viewed as a particular case of a multivaluational function. Under the multivaluational approach, classical logic can be viewed as based exactly on *one* initial truth value, namely on truth. That is, classical logic presupposes the *monism of truth*. Indeed, all the (classical) logical concepts can be formulated by using only the notion of truth (together with metalanguage connectives) with no mention of falsity. However, to have a logic, we need something *opposite* to truth, i.e. *non-truth*, and the multivaluational function produces such an element, this is again \emptyset . We can call this new element ‘falsity’, although this classical falsity is not an autonomous notion by itself, but completely depends on the notion of truth, having also one (negative) feature — not to be true. The generalized truth value space is in this case just the Boolean algebra of two elements, a *monolattice* sometimes called Lattice TV (cf. [40, p. 171]). Using the notation of the present paper this lattice can be named TWO_1 .

The next step is to take falsity seriously and consider it an independent notion on its own right. This step is made in the third row. The base of a generalized truth value space of this row contains exactly two elements — truth and falsity, and the multivaluational function gives us Belnap’s ‘useful four-valued logic’, which is in effect *relevance logic*. The corresponding algebraic structure is a bilattice $FOUR_2$.

Thus, the number n of elements in the base of a generalized truth value space is a semantical analogue for the geometrical notion of a ‘dimension’. However, algebraically the nature of the corresponding lattice is determined not by this number, but by the number of the ordering relations that we single out in this lattice. Therefore both the fourth and fifth rows in Table 6 represent trilattices ($EIGHT_3$ and $SIXTEEN_3$), although we have here 3- and 4-based truth value spaces.

In lattice theory, one should not take too seriously the ‘geometrical nature’ of the diagram in Figure 5. Purely geometrically

It is a mistake to suppose the tesseract is bounded by its 24 squares. They form only a skeleton of the hypercube, just as the edges of a cube form its skeleton. A cube is bounded by square faces and a hypercube by cubical faces. [20, p. 45]

But from a lattice-theoretical standpoint the only entities of the diagram that matter are just points (representing the elements of the lattice) and lines (showing how the points are connected by the ordering relations). Cubes and squares are not the structure-forming entities of $SIXTEEN_3$, although in itself they can be sublattices of the general lattice.

Thus, we obtain a collection of perfect truth value spaces that can be presented as a hierarchy of ‘logical worlds’. Each world has specific properties summarized in Table 7. Some of these worlds are explored quite well, while others still are *terra incognita* and are waiting for their investigators.

We finish the paper by pointing out a possible generalization which may result from our approach. The idea is to introduce the notion of n -lattice, where n is the number of specific partial orderings each of which forms a lattice on a given set. Such a structure can have interesting applications in a theory of truth, when we wish to consider other possible char-

TABLE 7. A hierarchy of logical worlds

	<i>Base (n-Space)</i>	<i>Truth values</i>	<i>Structure</i>	<i>Paradigmatic lattice</i>	<i>Logic</i>
Hegel's world	0	1	none	none	none
Frege's world	1	2	monolattice	TWO_1	classical logic
Belnap's world	2	4	bilattice	$FOUR_2$	relevant logic
Heyting-Nelson's world	3	8	trilattice	$EIGHT_3$	constructive relevant logic
Generalized Heyting-Nelson's world	4	16	trilattice	$SIXTEEN_3$?

acterizations of a truth predicate and to analyse, e.g. the *pragmatic* conception of truth, the *coherence* conception of truth, etc.

Acknowledgements

The paper was written during a research visit of Yaroslav Shramko and Tatsutoshi Takenaka to Indiana University in the academic year 1999–2000. Yaroslav Shramko is grateful to the *Fulbright Program* for supporting his work on this paper with a research fellowship. Tatsutoshi Takenaka gratefully acknowledges the support of Japan Patent Office. We wish to thank the Departments of Philosophy and Computer Science and the School of Informatics (Indiana University) for creating excellent working conditions.

References

- [1] A. Almukdad and D. Nelson. Constructible falsity and inexact predicates, *Journal of Symbolic Logic*, **49**, 231–233, 1984.
- [2] O. Arieli and A. Avron. Reasoning with logical bilattices, *Journal of Logic, Language and Information*, **5**, 25–63, 1996.
- [3] A. R. Anderson and N. D. Belnap, Jr. *Entailment The Logic of Relevance and Necessity*, Vol. I, Princeton University Press, 1975.
- [4] A. R. Anderson, N. D. Belnap, Jr. and J. M. Dunn. *Entailment The Logic of Relevance and Necessity*, Vol. II, Princeton University Press, 1992.
- [5] N. D. Belnap, Jr. A useful four-valued logic, In *Modern Uses of Multiple-Valued Logic*, J. M. Dunn and G. Epstein, eds. pp. 8–37. D. Reidel Publishing Co., Dordrecht, 1977.
- [6] N. D. Belnap, Jr. How a computer should think. In *Contemporary Aspects of Philosophy*, G. Ryle. ed. pp. 30–55. Oriel Press Ltd, Stocksfield, 1977.
- [7] H. S. M. Coxeter. *Regular Polytopes*, 2nd edn. Macmillan, New York, 1963.
- [8] M. Dummett. *Elements of Intuitionism*, Clarendon Press, Oxford, 1977.

- [9] M. Dummett. Reply to Dag Prawitz. In *Michael Dummett: Contributions to Philosophy*, B. Taylor, ed. pp. 281–286. Nijhoff, The Hague, 1987.
- [10] M. Dummett. Reply to Prawitz. In *The Philosophy of Michael Dummett*, B. McGuinness and G. Olivery, eds. pp. 292–298. Kluwer Academic Publishers, Dordrecht, 1994.
- [11] J. M. Dunn. *The Algebra of Intensional Logics*. Doctoral dissertation. University of Pittsburgh, Ann Arbor (University Microfilms), 1966
- [12] J. M. Dunn. Intuitive semantics for first-degree entailment and ‘coupled trees’, *Philosophical Studies*, **29**, 149–168, 1976.
- [13] J. M. Dunn. A Comparative study of various model-theoretic treatments of negation: a history of formal negation. In *What is Negation?*, Applied Logic Series, 13, D. M. Gabbay and H. Wansing, eds., pp. 23–51. Kluwer, 1999.
- [14] J. M. Dunn. Partiality and its dual, *Studia Logica*, **66**, 225–256, 2000.
- [15] J. M. Dunn and G. Restall. Relevance logic and entailment. In *Handbook of Philosophical Logic*, 2nd edn., Vol. 6, D. M. Gabbay and F. Guenther, eds. pp. 1–128. Kluwer, Dordrecht, 2001.
- [16] M. Fitting. Logic programming on a topological bilattice, *Fundamenta Informatica*, **11**, 209–218, 1988.
- [17] M. Fitting. Bilattices and the theory of truth, *Journal of Philosophical Logic*, **18**, 225–256, 1989.
- [18] M. Fitting. Kleene’s logic, generalized, *Journal of Logic and Computation*, **1**, 797–810, 1990.
- [19] G. Gargov. Knowledge, uncertainty and ignorance in logic: bilattices and beyond, *Journal of Applied Non-Classical Logics*, **9**, 1999.
- [20] M. Gardner. *Mathematical Carnival*, Alfred A. Knopf, New York, 1975
- [21] M. Ginsberg. Multi-valued logics, in: *Proceedings of AAAI-86, Fifth National Conference on Artificial Intelligence*, pp. 243–247. Morgan Kaufman Publishers, Los Altos, 1986.
- [22] M. Ginsberg. Multivalued logics: a uniform approach to reasoning in AI, *Computer Intelligence*, **4**, 256–316, 1988.
- [23] N. Goodman. *Fact, Fiction and Forecast*, 2nd edn., The Boobs-Merrill Co., Indianapolis, 1965.
- [24] A. Grzegorzczk. A philosophically plausible formal interpretation of intuitionistic logic, *Indagationes Mathematicae*, **26**, 596–601, 1964.
- [25] A. Heyting. A. Die formalen Regeln der intuitionistischen Logik. In *Sitzungsberichte der Preussischen Akademie der Wissenschaften, Physikalisch-Mathematische Klasse*, pp. 42–56. Berlin, 1930.
- [26] A. Heyting. *Intuitionism an Introduction*. Studies in Logic and Foundations of Mathematics, Amsterdam, 1956.
- [27] A. Heyting. La conception intuitionniste de la logique, *Les études philosophiques II*, pp. 226–233, 1956.
- [28] I. Johansson. Der Minimalkalkül, ein reduzierter intuitionistischer Formalismus, *Compositio Math.*, **4**, 119–136, , 1936.
- [29] A. S. Karpenko. Truth values: what they are? (in Russian). In *Investigations in Non-classical Logics*, V. A. Smirnov, ed. pp. 38–53. ‘Nauka’, Moscow, 1989.
- [30] L. V. S. Lakshmanan and F. Sadri. Probabilistic deductive databases. In *Proceedings of 1994 International Logic Programming Symposium*, M. Bruynooghe, ed. pp. 254–268, MIT Press, 1994.
- [31] E. G. K. López-Escobar. Refutability and elementary number theory, *Indagationes Mathematicae*, **34**, 362–374, 1972.
- [32] A. A. Markov. An approach to constructive mathematical logic. In *Logic, Methodology and Philosophy of Science III*, B. van Rootselaar and J. F. Staal, eds., pp. 283–294. North-Holland Publishing Company, Amsterdam, 1968.
- [33] D. Nelson. Constructible falsity, *Journal of Symbolic Logic*, **14**, 16–26, 1949.
- [34] D. Prawitz. Intuitionistic logic: a philosophical challenge. In *Logic and Philosophy*, G. H. von Wright, ed., pp. 1–10. Nijhoff, The Hague, 1980.
- [35] D. Prawitz. Dummett on the a theory of meaning and its impact on logic. In *Michael Dummett: Contributions to Philosophy*, B. Taylor, ed., pp. 117–165. Nijhoff, The Hague, 1987.
- [36] D. Prawitz. Meaning theory and anti-realism. In *The Philosophy of Michael Dummett*, B. McGuinness and G. Olivery, eds., pp. 79–89. Kluwer Academic Publishers, Dordrecht, 1994.
- [37] A. P. Pynko. Funktional completeness and axiomatizability within Belnap’s four-valued logic and its expansions, *Journal of Applied Non-Classical Logics*, **9**, 61–105, 1999.
- [38] A. P. Pynko. Regular bilattices, *Journal of Applied Non-Classical Logics*, **10**, 93–111, 2000.
- [39] W. Rabinowicz. Intuitionistic truth, *Journal of Philosophical Logic*, **14**, 1985, pp. 191–228
- [40] G. Restall. *An Introduction to Substructural Logics*, Routledge, London, 2000

- [41] D. Scott. Models for various type-free calculi. In *Logic, Methodology and Philosophy of Science*, vol. IV, P. Suppes, E. Nagel and A. Tarski, eds., pp. 157–187. North-Holland, Amsterdam, 1973.
- [42] A. Schöter. Evidential bilattice logic and lexical inference, *Journal of Logic, Language and Information*, **5**, 65–105, 1996.
- [43] Y. Shramko. On the problem of relevant entailment for intuitionistic logic (in Russian). In *Logico-Philosophical Investigations*, Vol. 1, pp. 165–175. Moscow, 1989.
- [44] Y. Shramko. A philosophically plausible modified Grzegorczyk semantics for first-degree intuitionistic entailment, *Logique et Analyse*, **161-162-163**, 167–188, 1998.
- [45] Y. Shramko. *Intuitionismus und Relevanz*. Logos-Verlag, Berlin, 1999.
- [46] Y. Shramko. American plan for intuitionistic logic 1: an intuitive background. In *The Logica Yearbook 1999*, T. Childers, ed., pp. 53–64. Filosofia, Prague, 2000.
- [47] Y. Shramko. American plan for intuitionistic logic 2: generalized Kripke models (in Russian), *Logical Studies*, No 5, 2000. (Online Journal, ISBN 5-85593-141-2, <http://www.logic.ru/LogStud/05/LS5.html>)
- [48] R. Thomason. A semantical study of constructive falsity, *Zeitschrift für mathematische Logik und Grundlagen der Mathematik*, **15**, 247–257, 1969.
- [49] W. P. Van Stigt. *Brouwer's Intuitionism*, North-Holland, Amsterdam, 1990.
- [50] H. Wansing. *The Logic of Information Structures*, Lectures Notes in AI, 681, Berlin, Springer-Verlag, 1993.

Received 10 March 2001