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ENTAILMENT RELATIONS AND/AS TRUTH VALUES

1. Introduction

It has been emphasized by Hiroakira Ono, Petr Hájek, and other logicians that there exists a close relationship between substructural and many-valued logics, see, for example, [11], [13], [18]. This relationship has many aspects, and in the present paper, we take the prominent substructural logic of first-degree entailment as a starting point for making some observations concerning many-valuedness and entailment.

Semantically, logic as a science of correct reasoning and valid arguments concentrates largely around the concept of entailment and investigations of its various aspects. Sometimes it is even said that this concept is *fundamental* to logic (see, e.g., [7], p. 55). It is therefore no wonder that entailment most typically is defined through another (philosophically maybe even more) fundamental concept—the one of truth. Namely, according to a standard understanding, sentence A entails sentence B if and only if B is true whenever A is true. Note that if the underlying context is classical, and falsity is interpreted simply as the absence of truth (as non-truth) and truth as the absence of falsity (as non-falsity), then this is just another way of saying that B is not false whenever A is not false. Two other popular ways to express essentially the same idea is to say that always either A is false or B is true, or that it is impossible that A is true and B is false.

Following Frege, truth and falsity are represented in logical semantics by the corresponding truth values, and in classical logic the only truth values that play a role in semantical constructions are the two truth values "the true" (T) and "the false" (F). The picture becomes more complex as soon as one brings into play the idea of a many-valued logic. It appears that the mere multiplication of *truth values* not only affords room for defining various *entailment relations*, but can also affect some of their essential characteristics. In this note we intend to show that the interconnection between these central logical notions, the notion of a truth value and the notion of entailment, is even more intimate than the connection emerging from a mutual interaction of their properties. In some cases it is possible to draw a strong analogy between them, namely to interpret entailment relations *as* a kind of truth values, and such an interpretation seems to be both natural and promising.

2. Three- and four-valued logics: matrices and lattices

Let $\mathcal{L}$ be a language in a denumerable set of sentence letters and a finite non-empty set of finitary sentential connectives $\mathcal{C} = \{c_1, \ldots, c_m\}$. The standard way of presenting a many-valued logic based on $\mathcal{L}$ is by means of an (n-valued) *logical matrix* which is defined as a structure $\langle \mathcal{V}, \mathcal{D}, \{f_c : c \in \mathcal{C}\}\rangle$, where $\mathcal{V}$ is a non-empty set of truth values (or just values) of cardinality $n \geq 2$, $\mathcal{D}$ is a non-empty proper subset of $\mathcal{V}$, and every f_c is a function on $\mathcal{V}$ with the same arity as c. With every logical matrix we associate a valuation function v that assigns elements of $\mathcal{V}$ to every sentence from $\mathcal{L}$. Elements of $\mathcal{D}$ are usually called *designated values*, and the set $\mathcal{D}$ is conceived as a generalization of the classical truth value T. Correspondingly, a natural generalization of classical entailment stipulates a definition of a many-valued entailment as a relation that under each valuation v preserves membership in the set of designated values from the premise to the conclusion:[1]

$$A \models B \text{ iff } \forall v : v(A) \in \mathcal{D} \Rightarrow v(B) \in \mathcal{D} \tag{1}$$

As the name suggests, a many-valued logic explicitly deals with *many* values. "Many" means here "more than two", for historically such logics

[1]For the sake of simplicity we consider entailment as a relation between (single) formulas having in mind its possible generalization to sets of formulas.

emerge as an alternative to classical logic, which is two-valued. Thus, even one additional value (extending the set of classical truth values $\{T, F\}$) gives rise to a many-valued logic, and indeed the first such logic constructed by Łukasiewicz as early as in 1918 (see [14] and [15]) was exactly three-valued. Following [6], p. 11 let us generally label the third value with I (as in a sense *intermediate* between truth and falsity) leaving thus room for its various concrete interpretations.

We will be especially interested in two other three-valued systems: Kleene's (strong) "logic of indeterminacy" K_3 and Priest's "logic of paradox" P_3. If we restrict ourselves to a non-implicational language, then K_3 is determined by the *Kleene matrix* $\mathsf{K}_3 = \langle\{T, I, F\}, \{T\}, \{f_c : c \in \{\sim, \wedge, \vee\}\}\rangle$, where the functions f_c are defined as follows:

$f_\sim$	
T	F
I	I
F	T

$f_\wedge$	T	I	F
T	T	I	F
I	I	I	F
F	F	F	F

$f_\vee$	T	I	F
T	T	T	T
I	T	I	I
F	T	I	F

The *Priest matrix* P_3 differs from K_3 only in that $\mathcal{D} = \{T, I\}$. Entailment in K_3 as well as in P_3 is defined by means of (1).

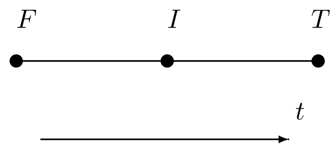

Figure 1: Lattice *THREE*

It is also well-known that the truth values of both Kleene's and Priest's logic can be ordered to form a lattice ($THREE$), which is diagrammed in Figure 1. Here T, I and F are ordered by means of a so-called *truth order* ($\leq_t$) so that the intermediate value I is "more true" than F, but "less true" than T. The operations of meet and join with respect to $\leq_t$ are exactly the functions $f_\wedge$ and $f_\vee$ above, and $f_\sim$ is just the inversion of this order.

There are natural intuitive interpretations of I in K_3 and in P_3 as the *underdetermined* and the *overdetermined* value respectively (a truth-value gap and a truth-value glut). Formally these interpretations can be modeled by presenting the values as certain subsets of the set of classical truth values $\{T, F\}$. Then T turns into $\mathbf{T} = \{T\}$ (understood as "true only"), F into $\mathbf{F}$

$= \{F\}$ ("false only"), I is interpreted in K_3 as $\mathbf{N} = \{\} = \emptyset$ ("*neither* true nor false"), and in P_3 as $\mathbf{B} = \{T, F\}$ ("*both* true and false").

If one combines all these new values into a joint framework, one obtains the four-valued logic B_4 introduced by Dunn and Belnap (see [2], [3], [5]) also known as the logic of first-degree entailment, or the logic of "tautological entailment" [1], § 15.2. This logic is determined by the *Belnap matrix* $\mathsf{B}_4 = \langle\{\mathbf{N}, \mathbf{T}, \mathbf{F}, \mathbf{B}\}, \{\mathbf{T}, \mathbf{B}\}, \{f_c : c \in \{\sim, \wedge, \vee\}\}\rangle$, where the functions f_c are defined as follows:

$f_\sim$	
T	**F**
B	**B**
N	**N**
F	**T**

$f_\wedge$	**T**	**B**	**N**	**F**
T	**T**	**B**	**N**	**F**
B	**B**	**B**	**F**	**F**
N	**N**	**F**	**N**	**F**
F	**F**	**F**	**F**	**F**

$f_\vee$	**T**	**B**	**N**	**F**
T	**T**	**T**	**T**	**T**
B	**T**	**B**	**T**	**B**
N	**T**	**T**	**N**	**N**
F	**T**	**B**	**N**	**F**

Definition (1) applied to the Belnap matrix determines the entailment relation of B_4.

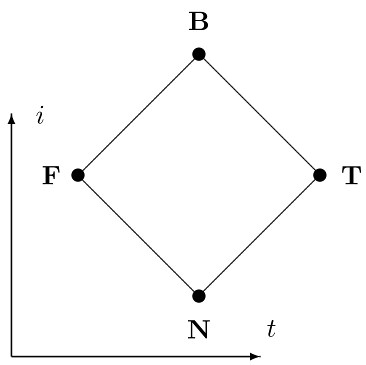

Figure 2: The bilattice $FOUR_2$

The set of truth values $\{\mathbf{N}, \mathbf{T}, \mathbf{F}, \mathbf{B}\}$ constitutes a specific algebraic structure – the *bilattice* $FOUR_2$ presented in Figure 2. The bilattice is equipped with *two* partial orderings: the information order ($\leq_i$) and the truth order ($\leq_t$). The relation $\leq_i$ is said to order the values under consideration according to the information they give concerning a formula to which they are assigned, and it is defined just as set-inclusion. The relation $\leq_t$ organizes the values concerning the amount of truth that is assigned by them.

Lattice meet and join with respect to $\leq_t$ coincide with the functions $f_\wedge$ and $f_\vee$ in the Belnap matrix B_4, and $f_\sim$ turns out to be the truth order inversion.

$FOUR_2$ is also of interest in that it allows another way of defining entailment, cf. [20], [22], [24]. Namely, one can consider this relation as expressing agreement with the truth order. That is:

$$A \models B \text{ iff } \forall v : v(A) \leq_t v(B) \tag{2}$$

It is very well known (see, e.g., [8], [21]) that (1) in B_4 and (2) in $FOUR_2$ define one and the same relation.

3. Antidesignated values, q-matrices and entailment relations

In any logical matrix the initial set of values is actually divided into two parts: those values that are designated and those that are not. While it seems natural to construe the set of designated values as a generalization of the classical truth value T, it would not always be adequate to interpret the other part (the set of non-designated values) as a generalization of the classical truth value F. The point is that in a many-valued logic, unlike in classical logic, "non-true" not always means "false" (cf., e.g., the above interpretation of Kleene's logic, where sentences can be neither true nor false).

In the literature on many-valued logic it is sometimes proposed to consider a set of *antidesignated* values which not obligatorily constitute the complement of the set of designated values (see, e.g., [12], [19]). The set of antidesignated values can be regarded as representing a generalized concept of falsity. Malinowski in [16] and [17] embodies this proposal by introducing the concept of an n-valued q-matrix (*quasi*-matrix) which is a structure $\langle \mathcal{V}, \mathcal{D}^+, \mathcal{D}^-, \{f_c : c \in \mathcal{C}\}\rangle$, where $\mathcal{V}$ is a non-empty set of values with at least two elements, $\mathcal{D}^+$ (the set of designated values) and $\mathcal{D}^-$ (the set of antidesignated values) are *disjoint* non-empty proper subsets of $\mathcal{V}$, and every f_c is a function on $\mathcal{V}$ with the same arity as c. In [22] it is argued that it is sometimes quite reasonable to admit $\mathcal{D}^+ \cap \mathcal{D}^- \neq \emptyset$, and in [23] we consider generalized q-matrices where it is not required that $\mathcal{D}^+ \cap \mathcal{D}^- = \emptyset$.

Now, it turns out that if the set of truth values is *not* dichotomized, but in fact trisected (or even more), this has a significant impact on the very concept of logical entailment. For example, since "truth" ($\mathcal{D}^+$) does not generally coincide with "non-falsity" (the complement of $\mathcal{D}^-$) any more, the expressions "B is true whenever A is true" and "B is not false whenever A is not false" need no longer mean the same. Moreover, the other usual characterizations of entailment: "in any case either A is false or B is true" and "it is impossible that A is true and B false", also become non-equivalent. Hence, the relations determined by these conditions may also be different.

Let us approach this subject in a systematic way. Granting that sets of designated and antidesignated values are not obligatorily complementations of each other, in addition to the simple preservation of truth and the simple preservation of non-falsity from (in the present case) the single premise to the single conclusion, there come to mind at least two other notions of entailment based on an obvious interplay between $\mathcal{D}^+$ and $\mathcal{D}^-$ which have a clear intuitive appeal (although these new notions are *not* defined in terms of preserving membership in some subset of the set of truth values $\mathcal{V}$). That is, we obtain the following four primitive definitions:[2]

$$A \models_t B \text{ iff } \forall v : v(A) \in \mathcal{D}^+ \Rightarrow v(B) \in \mathcal{D}^+ \tag{3}$$

$$A \models_f B \text{ iff } \forall v : v(A) \notin \mathcal{D}^- \Rightarrow v(B) \notin \mathcal{D}^- \tag{4}$$

$$A \models_q B \text{ iff } \forall v : v(A) \notin \mathcal{D}^- \Rightarrow v(B) \in \mathcal{D}^+ \tag{5}$$

$$A \models_p B \text{ iff } \forall v : v(A) \in \mathcal{D}^+ \Rightarrow v(B) \notin \mathcal{D}^- \tag{6}$$

We will refer to these relations as t-entailment, f-entailment, q-entailment and p-entailment, correspondingly. Whereas t-entailment is the standard truth-preserving relation, f-entailment incorporates the idea of non-falsity preservation (cf. [6], p. 10). The relation of q-entailment can be seen as reflecting a reasoning from hypotheses (understood as statements that merely are taken to be non-refuted). This relation has been introduced by Malinowski (together with the underlying concept of a q-matrix) in order to define a notion of entailment which depends on both $\mathcal{D}^+$ and $\mathcal{D}^-$, and to provide thereby a counterexample to Suszko's Thesis (the claim that every many-valued logic can be characterized by a two-valued semantics).

[2] We say "primitive", having in mind the possibility of considering further definitions obtained by combining the original conditions.

And p-entailment (p for "plausibility") has been studied by Frankowski [9], [10], who tried to explicate "reasonings wherein the degree of strength of the conclusion (i.e. the conviction it is true) is smaller then that of the premisses" [9], p. 41. Usually q-entailment and p-entailment are firmly associated with the corresponding q- and p-matrices, but here we operate more generally, considering the definitions of q-entailment and p-entailment as such. The idea is that these definitions may give rise to various concrete entailment relations when brought to the context of a concrete matrix.

And indeed, within a certain generalized four-valued setting, for example, some of these relations turn out to be undistinguishable. (In what follows, when we write $\models_x$ as an alone standing sign, we mean the respective entailment relation, i.e., the set of pairs (A, B) such that $A \models_x B$.)

PROPOSITION 1. *Let the Belnap generalized q-matrix* B_4^* *be the structure* $\langle\{\mathbf{N}, \mathbf{T}, \mathbf{F}, \mathbf{B}\}, \{\mathbf{T}, \mathbf{B}\}, \{\mathbf{F}, \mathbf{B}\}, \{f_c : c \in \{\sim, \wedge, \vee\}\}\rangle$ *(functions* f_c *being defined as in the usual Belnap matrix), and let us introduce on this structure the above four entailment relations. Then* $\models_t\, = \,\models_f$ *and* $\models_q\, = \,\models_p$*. Moreover,* $\models_t$ *(alias* $\models_f$*)* $\neq\, \models_p$ *(alias* $\models_q$*).*

PROOF. Dunn has shown that in the logic of first degree entailment $\models_t\, = \,\models_f$ (see Proposition 4 in [6], p. 10). He defined for every valuation v its dual v^*, such that for any formula A: $v(A) = \mathbf{T} \Leftrightarrow v^*(A) = \mathbf{T}$; $v(A) = \mathbf{F} \Leftrightarrow v^*(A) = \mathbf{F}$; $v(A) = \mathbf{B} \Leftrightarrow v^*(A) = \mathbf{N}$; $v(A) = \mathbf{N} \Leftrightarrow v^*(A) = \mathbf{B}$. It is not difficult to see that in B_4^* such a dual valuation does exist for every v. Note, that $v^{**} = v$.

Now, we show that $\models_p\, \subseteq\, \models_q$. Let $\forall v : v(A) = \mathbf{T}$ or $v(A) = \mathbf{B} \Rightarrow v(B) \neq \mathbf{F}$ and $v(B) \neq \mathbf{B}$. Assume $v(A) \neq \mathbf{F}$ and $v(A) \neq \mathbf{B}$. Then $v(A) = \mathbf{T}$ or $v(A) = \mathbf{N}$. Hence, $v^*(A) = \mathbf{T}$ or $v^*(A) = \mathbf{B}$. So, $v^*(B) \neq \mathbf{F}$ and $v^*(B) \neq \mathbf{B}$, and consequently, $v(B) \neq \mathbf{F}$ and $v(B) \neq \mathbf{N}$. Therefore $v(B) = \mathbf{T}$ or $v(B) = \mathbf{B}$.

Next, we show that $\models_q\, \subseteq\, \models_p$. First, observe that if $A \models_q B$, then there is no valuation v such that $v(A) = \mathbf{T}$ and $v(B) = \mathbf{B}$. Indeed, assume there exists such valuation v. Then $v^*(A) = \mathbf{T}$ and $v^*(B) = \mathbf{N}$. It follows that $v^*(B) = \mathbf{T}$ or $v^*(B) = \mathbf{B}$, a contradiction. Now, let $\forall v : v(A) \neq \mathbf{F}$ and $v(A) \neq \mathbf{B} \Rightarrow v(B) = \mathbf{T}$ or $v(B) = \mathbf{B}$. Assume $v(A) = \mathbf{T}$ or $v(A) = \mathbf{B}$. If $v(A) = \mathbf{T}$, then $v(B) = \mathbf{T}$ or $v(B) = \mathbf{B}$. But, the second case is impossible, and hence, $v(B) = \mathbf{T}$, i.e., $v(B) \neq \mathbf{F}$ and $v(B) \neq \mathbf{B}$. If $v(A) = \mathbf{B}$, then $v^*(A) = \mathbf{N}$ and thus, $v^*(B) = \mathbf{T}$ or $v^*(B) = \mathbf{B}$. Hence,

$v(B) = \mathbf{T}$ or $v(B) = \mathbf{N}$. In both cases, again, $v(B) \neq \mathbf{F}$ and $v(B) \neq \mathbf{B}$, which is the required result.

Thus, in the generalized q-matrix B_4^* the four entailment relations merge into two. To show that these two relations are distinct, we observe that in B_4^*: $A \models_t A$ (as well as $A \models_f A$), but $A \not\models_p A$ (and also $A \not\models_q A$). (Otherwise we would have $v(A) \in \mathcal{D}^+ \Rightarrow v(A) \notin \mathcal{D}^-$ and $v(A) \notin \mathcal{D}^- \Rightarrow v(A) \in \mathcal{D}^+$, and this would mean that $\mathcal{D}^+ \cap \mathcal{D}^- = \emptyset$, which is not the case.)[3] □

4. Orderings on entailment relations

In our particular example of a *generalized four-valued* q-matrix, it turned out that $\models_t\ =\ \models_f$ and $\models_q\ =\ \models_p$. Interestingly, in a non-generalized setting the picture may become even more complex. Let the (ordinary) *Kleene-Priest q-matrix* KP_3^* be the structure $\langle\{T, I, F\}, \{T\}, \{F\}, \{f_c : c \in \{\sim, \wedge, \vee\}\}\rangle$, where the functions f_c are defined as in K_3 and P_3, and let us consider the entailment relations defined by (3)–(6) with respect to this matrix. Then, these relations are *all distinct*, $\models_t$ is the entailment relation of Kleene's logic, and $\models_f$ corresponds to the entailment of Priest's logic (cf. [6]). Moreover, the following proposition exposes some simple facts about the interconnections between the entailments:

Proposition 2. *In the Kleene-Priest q-matrix: (i)* $\models_q\ \subseteq\ \models_t$*; (ii)* $\models_q\ \subseteq\ \models_f$*; (iii)* $\models_t\ \subseteq\ \models_p$*; (iv)* $\models_f\ \subseteq\ \models_p$*; (v)* $\models_q\ \subset\ \models_t \cap \models_f$.

Proof. (i) Let $\forall v : v(A) \notin \mathcal{D}^- \Rightarrow v(B) \in \mathcal{D}^+$. Assume $\exists v : v(A) \in \mathcal{D}^+$ and $v(B) \notin \mathcal{D}^+$. Then $v(A) \in \mathcal{D}^-$, a contradiction with the condition $\mathcal{D}^+ \cap \mathcal{D}^- = \emptyset$.

(ii)–(iv) The proof is analogous.

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[3]Incidentally, this observation shows that in the context of B_4^*, $\models_p$ and $\models_q$ are not Tarskian *even if* these two relations coincide. (Recall that an entailment relation is Tarskian iff it satisfies *reflexivity*, *monotonicity* and *cut*.) In this respect generalized q-matrices differ from non-generalized q-matrices since, as Frankowski has shown in [10], p. 198, if a relation defined on a non-generalized q-matrix is both a p- and a q-entailment relation, then it *is* Tarskian.

(v) Clauses (i) and (ii) state that $\models_q \subseteq \models_t \cap \models_f$. That the set-inclusion is proper follows from the fact that in any proper q-matrix[4] (and hence in KP_3^*): $A \not\models_q A$, whereas both $A \models_t A$ and $A \models_f A$.[5] □

Facts (i)–(iv) are mentioned by Devyatkin in [4]. Note that they actually reveal that in KP_3^* our four entailment relations are ordered by means of $\subseteq$ to form a lattice with $\models_q$ as the bottom and $\models_p$ as the top. At this place an analogy comes to mind with the information ordering in bilattice $FOUR_2$ which is defined exactly as set-inclusion. It turns out that it is not the only possible analogy one can draw here. Let $\models_x^+$ stand for the part of $\models_x$ which only preserves designated values and $\models_x^-$ for the part of $\models_x$ that only preserves non-antidesignated values:

$$\models_x^+ := \{(A, B) \in \models_x \mid \begin{array}{l}(\forall v : v(A) \in \mathcal{D}^+ \Rightarrow v(B) \in \mathcal{D}^+ \text{ and} \\ \exists v : v(A) \notin \mathcal{D}^- \text{ and } v(B) \in \mathcal{D}^-)\}\end{array} \tag{7}$$

$$\models_x^- := \{(A, B) \in \models_x \mid \begin{array}{l}(\forall v : v(A) \notin \mathcal{D}^- \Rightarrow v(B) \notin \mathcal{D}^- \text{ and} \\ \exists v : v(A) \in \mathcal{D}^+ \text{ and } v(B) \notin \mathcal{D}^+)\}\end{array} \tag{8}$$

Intuitively $\models_x^+$ and $\models_x^-$ can be seen as representing correspondingly the "pure truth content" and the "pure falsity content" of $\models_x$. One can define then a "truth order" between the entailment relations:

$$\models_x \leq_t \models_y \text{ iff } \models_x^+ \subseteq \models_y^+ \text{ and } \models_y^- \subseteq \models_x^- \tag{9}$$

The following proposition shows how the four entailment relations can be organized into a logical lattice with $\models_f$ as the bottom and $\models_t$ as the top:

Proposition 3. *In the Kleene-Priest q-matrix: (i)* $\models_f \leq_t \models_q$*; (ii)* $\models_f \leq_t \models_p$*; (iii)* $\models_q \leq_t \models_t$*; (iv)* $\models_p \leq_t \models_t$*.*

Proof. As a direct consequence of Proposition 2 (i–iv) and Definitions (7) and (8) we obtain: $\models_t^+ = \models_t \setminus \models_f$; $\models_f^+ = \emptyset$; $\models_q^+ = \emptyset$; $\models_p^+ = \models_t^+$,

[4] We call a q-matrix *proper* iff $\mathcal{D}^+ \cup \mathcal{D}^- \neq \mathcal{V}$ holds for it.

[5] We leave as an open question whether in KP_3^* $\models_p \subseteq \models_t \cup \models_f$. An anonymous referee turned our attention to the fact that it is not difficult to define a particular q-matrix in which, e.g., $\models_p = \models_t = \models_f$. But it is also possible to define a q-matrix such that $\models_t \cup \models_f \subset \models_p$. Indeed, add to KP_3^* a unary truth function $f_\#$ such that $f_\#(T) = I$ and $f_\#(I) = F$. Then obviously $A \models_p \#A$, but $A \not\models_t \#A$ and $A \not\models_f \#A$.

as well as $\models_f^- = \models_f \setminus \models_t$; $\models_t^- = \emptyset$; $\models_q^- = \emptyset$; $\models_p^- = \models_f^-$. Hence the proposition. □

By combining Propositions 2 and 3 we immediately see that the entailment relations defined in the q-matrix KP_3^* constitute a structure isomorphic to the bilattice $FOUR_2$, where $\models_t$ is analogous to $\mathbf{T}$, $\models_f$ plays the role of $\mathbf{F}$, $\models_q$ stands for $\mathbf{N}$, and $\models_p$ is like $\mathbf{B}$. In this way we obtain another representation of a four-valued logic whose values are formed by the entailment relations defined on the basis of a three-valued quasi-matrix.

5. Concluding remarks

Throughout this note, we constantly had in view the concrete q-matrix KP_3^* and the concrete generalized q-matrix B_4^*. Nevertheless, one can observe that Proposition 1 can easily be extended to *any* generalized q-matrix, provided it is proper,[6] and its truth functions allow dualization, i.e., for any valuation v there exists a valuation v^* subject to the following conditions (for any formula A):

$$v(A) \in \mathcal{D}^+ \text{ and } v(A) \notin \mathcal{D}^- \Leftrightarrow v^*(A) \in \mathcal{D}^+ \text{ and } v^*(A) \notin \mathcal{D}^- \tag{10}$$

$$v(A) \notin \mathcal{D}^+ \text{ and } v(A) \in \mathcal{D}^- \Leftrightarrow v^*(A) \notin \mathcal{D}^+ \text{ and } v^*(A) \in \mathcal{D}^- \tag{11}$$

$$v(A) \in \mathcal{D}^+ \text{ and } v(A) \in \mathcal{D}^- \Leftrightarrow v^*(A) \notin \mathcal{D}^+ \text{ and } v^*(A) \notin \mathcal{D}^- \tag{12}$$

$$v(A) \notin \mathcal{D}^+ \text{ and } v(A) \notin \mathcal{D}^- \Leftrightarrow v^*(A) \in \mathcal{D}^+ \text{ and } v^*(A) \in \mathcal{D}^- \tag{13}$$

The four truth values of B_4^* can generally be modeled by the following conditions: $v(A) = \mathbf{T}$ iff $v(A) \in \mathcal{D}^+$ and $v(A) \notin \mathcal{D}^-$; $v(A) = \mathbf{F}$ iff $v(A) \notin \mathcal{D}^+$ and $v(A) \in \mathcal{D}^-$; $v(A) = \mathbf{B}$ iff $v(A) \in \mathcal{D}^+$ and $v(A) \in \mathcal{D}^-$; $v(A) = \mathbf{N}$ iff $v(A) \notin \mathcal{D}^+$ and $v(A) \notin \mathcal{D}^-$. Then it is not difficult to rewrite the proof of Proposition 1 in a general form so that it holds for any proper generalized q-matrix (with dual valuations).

Note, however, that if a generalized q-matrix includes "non-dualizable" truth functions, then Proposition 1 fails. Consider, for example, the generalized q-matrix $\langle\{\mathbf{N}, \mathbf{T}, \mathbf{F}, \mathbf{B}\}, \{\mathbf{T}, \mathbf{B}\}, \{\mathbf{F}, \mathbf{B}\}, \{f_\bullet\}\rangle$, where the unary truth

[6] Again, a generalized q-matrix is *proper* iff it satisfies the conditions: $\mathcal{D}^+ \cup \mathcal{D}^- \neq \mathcal{V}$ and $\mathcal{D}^+ \cap \mathcal{D}^- \neq \emptyset$.

function $f_\bullet$ is defined as follows: $f_\bullet(\mathbf{B}) = \mathbf{B}$, $f_\bullet(\mathbf{T}) = \mathbf{T}$, $f_\bullet(\mathbf{F}) = \mathbf{F}$, and $f_\bullet(\mathbf{N}) = \mathbf{F}$. Obviously, $A \models_t \bullet A$ but $A \not\models_f \bullet A$ and $A \models_p \bullet A$ but $A \not\models_q \bullet A$.

Also Propositions 2 (i–iv) and 3 can be extended to *any* (non-generalized) q-matrix (and Proposition 2 (v) to any proper non-generalized q-matrix), for in the corresponding proofs no specific use is made of any concrete feature of KP_3^*.

Thus, the main result of this article can also be summarized as follows: it formulates a simple but fairly general method of constructing a generalized four-valued q-matrix by taking as its values the basic entailment relations defined on an arbitrary non-generalized q-matrix.

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