A Note on Two Ways of Defining a Many-valued Logic

Heinrich Wansing* Yaroslav Shramko

1. Introduction

One can find in the literature two different ways of defining a many-valued logic. The standard way of presenting such a logic is by singling out a non-empty set of designated values and defining entailment as the preservation of membership in this set of designated values from the premises to the conclusions. In other words, a propositional many-valued logic according to this approach is given by a matrix, a non-empty set $\mathcal{V}$ of truth values (alias truth degrees) containing at least two elements, a nonempty set $\mathcal{D} \subset \mathcal{V}$ of designated truth values, and a set of truth functions $\{f_c \mid c \in \mathcal{C}\}$, where $\mathcal{C}$ is a finite set of primitive finitary connectives, f_c and c having the same arity. It is well-known, however, that the truth values of certain many-valued logics constitute lattices, see, for example, (Rose, 1951). According to Arieli and Avron (Arieli & Avron, 1996, p. 25) it is even the case that "[w]hen using multiple-valued logics, it is usual to order the truth values in a lattice structure". Thus, one obtains another (a slightly less standard) way of defining a many-valued logic which proceeds by (i) defining a lattice order $\leq$ on the set $\mathcal{V}$ of truth values, (ii) interpreting logical operations as operations on the lattice $(\mathcal{V}, \leq)$, and (iii) stipulating that a formula A entails a formula B iff for every homomorphic valuation function v, $v(A) \leq v(B)$. The most well-known example of the latter approach is provided by the bilattice $FOUR_2$, see, for instance, (Arieli & Avron, 1996), (Fitting, 2004), (Ginsberg, 1988), which gives rise to the logic of first-degree entailment, also called the logic of "tautological entailment" (Anderson & Belnap, 1975, § 15.2) or Belnap's and Dunn's useful four-valued logic (Belnap, 1977a), (Belnap, 1977b), (Dunn, 1976). In $FOUR_2$ the lattice order used to define entailment is interpreted as

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a truth order on the set of generalized truth values $\mathbf{4} = \{\mathbf{N}, \mathbf{T}, \mathbf{F}, \mathbf{B}\}$, where $\mathbf{N} = \varnothing$, $\mathbf{T} = \{T\}$, $\mathbf{F} = \{F\}$, $\mathbf{B} = \{T, F\}$, and $\{T, F\}$ is the set of classical truth values *true* and *false*. First-degree entailment has a matrix presentation, and the coincidence of the matrix and the lattice presentation may be seen as mutual support for the naturalness of both first-degree entailment and the two ways of presenting this logic.

Nevertheless, the two ways of defining a many-valued logic are generally non-equivalent in the sense that not every matrix presentation gives rise to an equivalent lattice presentation. In this note we shall briefly recall the relevant facts concerning $FOUR_2$ (section 1), describe an example where the two approaches might, perhaps, be expected to lead to the same logic, and point out that this is not the case (section 2), and conclude with a short remark on uniformly defining implication in the lattice approach, which might be seen as an advantage of this *modus operandi* (section 3).

2. $FOUR_2$ and B_4

If we consider the set of truth values $\mathcal{V} = \mathbf{4}$, the bilattice $FOUR_2$ presented in Figure 1 is said to order the truth values under consideration according to the information they give concerning a formula to which they are assigned and concerning the amount of truth that is assigned by these values. The information order $\leq_i$ is just set-inclusion, and the truth order ($\leq_t$) is based on certain assumptions about truth, falsity, and non-informativeness. It is, for instance, assumed that the value $\{T, F\}$ ("both true and false") is less true than $\{T\}$ ("only true").

Conjunction $\wedge$ may be interpreted as lattice meet and disjunction $\vee$ as lattice join with respect to $\leq_t$. Negation $\sim$ may be interpreted as truth order inversion satisfying the double-negation laws. This interpretation is captured by the following tables for the corresponding truth functions:

$f_\sim$	
T	**F**
B	**B**
N	**N**
F	**T**

$f_\wedge$	**T**	**B**	**N**	**F**
T	**T**	**B**	**N**	**F**
B	**B**	**B**	**F**	**F**
N	**N**	**F**	**N**	**F**
F	**F**	**F**	**F**	**F**

$f_\vee$	**T**	**B**	**N**	**F**
T	**T**	**T**	**T**	**T**
B	**T**	**B**	**T**	**B**
N	**T**	**T**	**N**	**N**
F	**T**	**B**	**N**	**F**

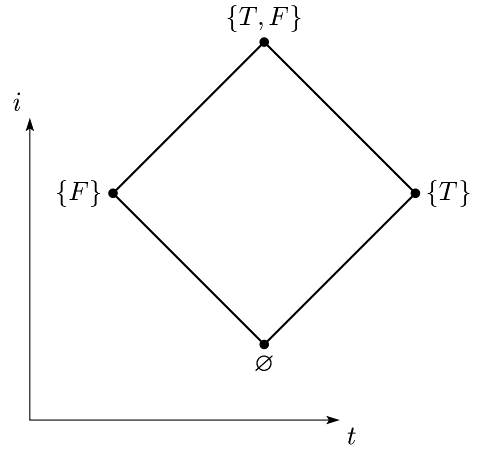

Figure 1. The bilattice $FOUR_2$

The relation $A \models_t^4 B$ between formulas A and B is defined by requiring that for every homomorphic valuation function v from the propositional language into **4** the following holds: $v(A) \leq_t v(B)$. It is well-known that $\models_t^4$ is exactly first-degree entailment logic.[1]

The relation $\models_t^4$ can also be described in terms of the preservation of designated truth values. Consider the matrix

$$B_4 = \langle\{\mathbf{N}, \mathbf{T}, \mathbf{F}, \mathbf{B}\}, \{\mathbf{T}, \mathbf{B}\}, \{f_c : c \in \{\sim, \wedge, \vee\}\}\rangle,$$

with its associated entailment relation $\models_4^+$ defined as:

$$A \models_4^+ B \text{ iff } \forall v\, (v(A) \in \mathcal{D} = \{\mathbf{T}, \mathbf{B}\} \text{ implies } v(B) \in \mathcal{D}).$$

The designated truth values are the values which contain the classical value T, and we have $\models_4^+ = \models_t^4$ (see (Font, 1997), (Shramko & Zaitsev, 2004)). (However, if we change the definition of designated values and put, e.g., $\mathcal{D} = \{\mathbf{T}\}$, the two ways of defining entailment do not result in one and the same relation. We obtain the *ex falso quodlibet* entailment $(A \wedge \sim A) \models_4^+ B$, for all formulas A and B.)

Furthermore, in (Wansing & Shramko, 2007) we introduce a so-called "separated 4-valued propositional logic" B_4* which makes use of *antides-*

[1] As we point out in (Shramko & Wansing, 2005), one could equally introduce in $FOUR_2$ a falsity order $\leq_f$ which turns out to be just the inversion of the truth order. Hence, the possible (and natural) definition of the entailment relation $\models_f^4$ which requires that $A \models_f^4 B$ iff for any valuation v: $v(B) \leq_f v(A)$ obviously results in the same relation as $\models_t^4$.

ignated values. The idea is to define a many-valued logic by introducing *two sets* of distinguished truth values, namely a set $\mathcal{D}^+$ of *designated* values associated with truth and another set $\mathcal{D}^-$ of *antidesignated* values associated with falsity. In the literature on many-valued logics such a distinction is well-known, see, (Gottwald, 1989), (Gottwald, 2001), (Malinowski, 1993), (Rescher, 1969). The distinction leaves room for values that are neither designated *nor* antidesignated and for values that are *both* designated *and* antidesignated. Malinowski (Malinowski, 1990), (Malinowski, 1993), (Malinowski, 1994), and Gottwald (Gottwald, 2001), however, impose the condition that $\mathcal{D}^+ \cap \mathcal{D}^- = \varnothing$. In (Wansing & Shramko, 2007) it is argued that it is not at all unreasonable to admit $\mathcal{D}^+ \cap \mathcal{D}^- \neq \varnothing$.

Malinowski calls any structure $\mathfrak{M} = \langle \mathcal{V}, \mathcal{D}^+, \mathcal{D}^-, \{f_c : c \in \mathcal{C}\}\rangle$, where $\mathcal{V}$ is a non-empty set with at least two elements, $\mathcal{D}^+$ and $\mathcal{D}^-$ are distinct non-empty proper subsets of $\mathcal{V}$ such that $\mathcal{D}^+ \cap \mathcal{D}^- = \varnothing$, and every f_c is a function on $\mathcal{V}$ with the same arity as c, an n-valued q-matrix (quasi-matrix). If it is not required that $\mathcal{D}^+ \cap \mathcal{D}^- \neq \varnothing$, we may talk of generalized q-matrices. A valuation function v in $\mathfrak{M}$ is a function from $\mathcal{L}$ into the set of truth degrees $\mathcal{V}$, and we may restrict our attention to valuations which satisfy the recursive conditions stated above.[2] Every q-matrix is an example of what is called a *ramified* matrix in (Wójcicki, 1988). In a ramified matrix there are finitely many distinguished sets of values $\mathcal{D}_1, \ldots, \mathcal{D}_j$.

The mentioned separated 4-valued propositional logic $B_4{}^*$ is given with the generalized q-matrix

$$\langle\{\mathbf{N}, \mathbf{T}, \mathbf{F}, \mathbf{B}\}, \{\mathbf{T}, \mathbf{B}\}, \{\mathbf{F}, \mathbf{B}\}, \{f_c : c \in \{\sim, \wedge, \vee\}\}\rangle.$$

The antidesignated truth values are the values which contain the classical value F, and the idea of antidesignated values naturally suggests the introduction of another entailment relation which is defined in terms of preservation of antidesignated values from the conclusion to the premise:

$$A \models_4^- B \text{ iff } \forall v(v(B) \in \mathcal{D}^- = \{\mathbf{F}, \mathbf{B}\} \text{ implies } v(A) \in \mathcal{D}^-).$$

Now, it is very well-known (see, e.g. (Dunn, 2000, p. 10)) that in $B_4{*} \models_4^+ = \models_4^-$. That is, in Belnap's and Dunn's four-valued logic not only the matrix and the lattice presentations turn out to be coincident, but so are the two

[2] Malinowski defines a kind of relation, called q-entailment, which depends on both sets $\mathcal{D}^+$ and $\mathcal{D}^-$. A q-matrix $\mathfrak{M} = \langle \mathcal{V}, \mathcal{D}^+, \mathcal{D}^-, \{f_c : c \in \mathcal{C}\}\rangle$ determines a q-entailment relation $\models_{\mathfrak{M}} \subseteq \mathcal{P}(\mathcal{L}) \times \mathcal{L}$ by defining $\Delta \models_{\mathfrak{M}} A$ iff for every valuation v in $\mathfrak{M}$, $v(\Delta) \cap \mathcal{D}^- = \varnothing$ implies $v(A) \in \mathcal{D}^+$, where $v(\Delta) = \{v(B) \mid B \in \Delta\}$. A q-entailment relation in general is not reflexive and does not admit of a reduction to a bivalent semantics.

entailment relations defined in terms of designated and antidesignated truth values.

A general question concerning an instantiation of the the matrix approach is: Given a set of truth values, how is the choice of the designated (and antidesignated) truth values justified? A corresponding question concerning the lattice approach is: Given a set of truth values, how is the definition of the lattice order(s) on this set justified? Arieli and Avron (Arieli & Avron, 1996, p. 29) explain that "[f]requently, in an algebraic treatment of the subject, the set of the designated values forms a filter, or even a prime (ultra-) filter, relative to some natural ordering of the truth values". As mentioned above, it is emphasized in (Shramko & Wansing, 2005, p. 128) that in the bilattice $FOUR_2$ truth and falsity are not dealt with as independent of each other, because it is assumed that F by itself is less true than T, and hence, in $FOUR_2$ $\{T, F\}$ is taken to be less true than $\{T\}$. The set of designated values of B_4 might be justified by pointing out that it is a prime filter with respect to $\leq_t$ in $FOUR_2$, but if the truth order $\leq_t$ itself is not convincingly justified, this criterion remains fairly technical. In (Shramko & Wansing, 2005) and (Shramko & Wansing, 2006) it is argued that if we move from the set **4** of generalized truth values to its powerset $\mathcal{P}(\mathbf{4}) = \mathbf{16}$, we may not only obtain a *truth* order on the underlying set of values that is better justified than the truth order of $FOUR_2$ but in addition also an independent and equally well-justified *falsity* order. Moreover, against the background of these separate truth and falsity orderings the definition of sets of designated values in terms of containment of T and in terms of of containment of F appears to be quite natural. Interestingly, in this case the two ways of defining a many-valued propositional logic do not converge.

3. Two natural 16-valued logics

Belnap's and Dunn's useful four-valued logic has been motivated by the idea of truth values encoding information concerning a given proposition, information which may be passed on to a computer. A generalization of this idea based on assuming that a so-informed computer may again pass on information to other recipients leads in a first step from **4** to **16** with the following 16 generalized truth values (where $\mathbf{A} = \mathbf{NFTB}$ stands for "all"):

1. $\mathbf{N} = \varnothing$,
2. $\boldsymbol{N} = \{\varnothing\}$,
3. $\boldsymbol{F} = \{\{F\}\}$,
4. $\boldsymbol{T} = \{\{T\}\}$,
5. $\boldsymbol{B} = \{\{F,T\}\}$,
6. $\mathbf{NF} = \{\varnothing, \{F\}\}$,
7. $\mathbf{NT} = \{\varnothing, \{T\}\}$,
8. $\mathbf{NB} = \{\varnothing, \{F,T\}\}$,
9. $\mathbf{FT} = \{\{F\}, \{T\}\}$,
10. $\mathbf{FB} = \{\{F\}, \{F,T\}\}$,
11. $\mathbf{TB} = \{\{T\}, \{F,T\}\}$,
12. $\mathbf{NFT} = \{\varnothing, \{F\}, \{T\}\}$,
13. $\mathbf{NFB} = \{\varnothing, \{F\}, \{F,T\}\}$,
14. $\mathbf{NTB} = \{\varnothing, \{T\}, \{F,T\}\}$,
15. $\mathbf{FTB} = \{\{F\}, \{T\}, \{F,T\}\}$,
16. $\mathbf{A} = \{\varnothing, \{T\}, \{F\}, \{F,T\}\}$,

Whereas in $FOUR_2$ the truth order is defined in terms of both T and F, on the set $\mathbf{16}$ separate truth and falsity orderings $\leq_t$ and $\leq_f$ can be isolated. In this 16-valued setting, truth and falsity are thereby treated as independent notions in their own right. For every x in $\mathbf{16}$ we first define the sets x^t, x^{-t}, x^f, and x^{-f} as follows:

$$x^t := \{y \in x \mid T \in y\}\,; \qquad x^{-t} := \{y \in x \mid T \notin y\}\,;$$
$$x^f := \{y \in x \mid F \in y\}\,; \qquad x^{-f} := \{y \in x \mid F \notin y\}\,.$$

For every x, y in $\mathbf{16}$ we then put:

1. $x \leq_i y$ iff $x \subseteq y$;
2. $x \leq_t y$ iff $x^t \subseteq y^t$ and $y^{-t} \subseteq x^{-t}$;
3. $x \leq_f y$ iff $x^f \subseteq y^f$ and $y^{-f} \subseteq x^{-f}$.

The truth order $\leq_t$ is thus defined by referring only to the classical value T, and $\leq_f$ is defined by referring only to F. Again, set-inclusion is interpreted as an information order. We obtain an algebraic structure that combines the three (complete) lattices $(\mathbf{16}, \leq_i)$, $(\mathbf{16}, \leq_t)$, and $(\mathbf{16}, \leq_f)$ into the *trilattice* $SIXTEEN_3 = (\mathbf{16}, \leq_i, \leq_t, \leq_f)$, see (Shramko & Wansing, 2005). $SIXTEEN_3$ is presented by a triple Hasse diagram in Figure 2, cf. also (Shramko, Dunn, & Takenaka, 2001).

Meets and joints exist in $SIXTEEN_3$ for all three partial orders. We will use $\sqcap$ and $\sqcup$ with the appropriate subscripts for these operations under the corresponding ordering relations. Since from the operations one can recover the relations, $SIXTEEN_3$ may also be represented as the structure $(\mathbf{16}, \sqcap_i, \sqcup_i, \sqcap_t, \sqcup_t, \sqcap_f, \sqcup_f)$. In what follows we shall focus on the "logical" operations $\sqcap_t$, $\sqcup_t$, $\sqcap_f$ and $\sqcup_f$. Since the relations $\leq_t$ and $\leq_f$ are treated on a par, the operations $\sqcap_t$ and $\sqcup_t$ are not privileged as

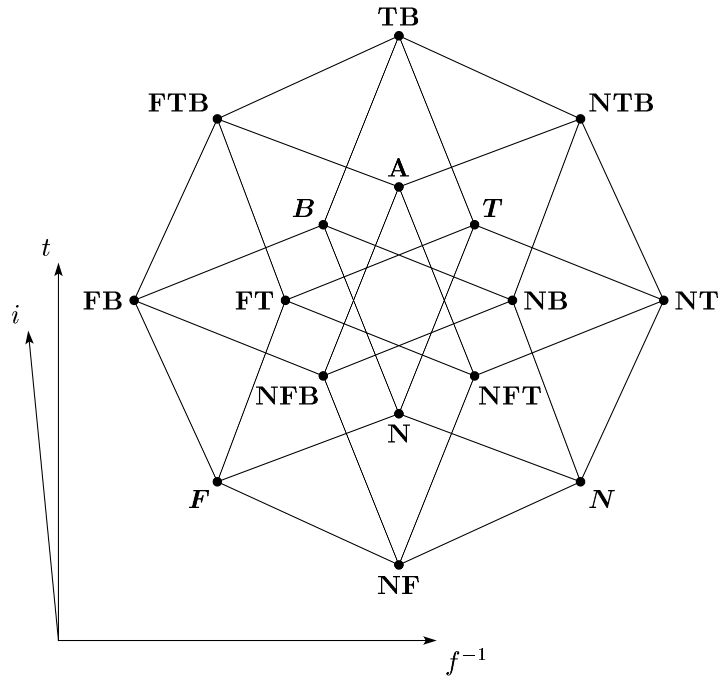

Figure 2. Trilattice $SIXTEEN_3$ (projection t - f^{-1})

interpretations of conjunction and disjunction. The operation $\sqcup_f$ may as well be regarded as a conjunction and $\sqcap_f$ as a disjunction. In other words, the logical vocabulary may be naturally considered to comprise a positive truth vocabulary and a negative falsity vocabulary. Also certain unary truth and falsity operations with natural negation-like properties are available in $SIXTEEN_3$. A unary operation $-_t$ ($-_f$) on $SIXTEEN_3$ is said to be a t-inversion (f-inversion) iff the following conditions are satisfied:

1. *t-inversion*$(-_t)$:
   (a) $a \leq_t b \Rightarrow -_t b \leq_t -_t a$;
   (b) $a \leq_f b \Rightarrow -_t a \leq_f -_t b$;
   (c) $a \leq_i b \Rightarrow -_t a \leq_i -_t b$;
   (d) $-_t -_t a = a$.

2. *f-inversion*$(-_f)$:
   (a) $a \leq_t b \Rightarrow -_f a \leq_t -_f b$;
   (b) $a \leq_f b \Rightarrow -_f b \leq_f -_f a$;
   (c) $a \leq_i b \Rightarrow -_f a \leq_i -_f b$;
   (d) $-_f -_f a = a$.

A t-inversion (f-inversion) thus inverts the truth (falsity) order, leaves the other orders untouched, and is period-two. In $SIXTEEN_3$ such operations are definable as shown in Table 1. The requirements that the information order $\leq_i$ is left untouched by the operations of t-inversion and f-inversion and that t-inversion (f-inversion) has no effect on $\leq_f$ ($\leq_t$) *are* satisfied by the operations $-_t$ and $-_f$ defined in Table 1, but these requirements might also be given up. If they are abandoned, the definition of t-inversion (f-inversion) refers only to the truth-order (falsity-order). What this suggests is that not only conjunction and disjunction, but also negation emerges in two versions, because $-_t$ and $-_f$ are both natural interpretations for a negation connective. Moreover, since $x \sqcap_t y \neq x \sqcup_f y$, $x \sqcup_t y \neq x \sqcap_f y$ and $-_t x \neq -_f x$, the two logical orderings $\leq_t$ and $\leq_f$ indeed give rise to pairs of *distinct* logical operations with the same arity.

a	$-_t a$	$-_f a$	a	$-_t a$	$-_f a$
N	**N**	**N**	**NB**	**FT**	**FT**
N	***T***	***F***	**FB**	**FB**	**NT**
F	***B***	***N***	**TB**	**NF**	**TB**
T	***N***	***B***	**NFT**	**NTB**	**NFB**
B	***F***	***T***	**NFB**	**FTB**	**NFT**
NF	**TB**	**NF**	**NTB**	**NFT**	**FTB**
NT	**NT**	**FB**	**FTB**	**NFB**	**NTB**
FT	**NB**	**NB**	**A**	**A**	**A**

Table 1. t- and f-inversions in $SIXTEEN_3$

A suitable syntax for the logic emerging from $SIXTEEN_3$ is given by a denumerable set of propositional variables and three propositional languages $\mathcal{L}_t$, $\mathcal{L}_f$, and $\mathcal{L}_{tf}$ based on this set. They are defined in Backus-Naur form as follows:

$$
\begin{aligned}
\mathcal{L}_t: &\quad A ::= p \mid \sim_t \mid \wedge_t \mid \vee_t \\
\mathcal{L}_f: &\quad A ::= p \mid \sim_f \mid \wedge_f \mid \vee_f \\
\mathcal{L}_{tf}: &\quad A ::= p \mid \sim_t \mid \sim_f \mid \wedge_t \mid \vee_t \mid \wedge_f \mid \vee_f
\end{aligned}
$$

Following the lattice approach to defining a many-valued logic, the logic of $SIXTEEN_3$ is semantically presented as a *bi-consequence system*, namely the structure $(\mathcal{L}_{tf}, \models_t, \models_f)$, where the two entailment relations $\models_t$ and $\models_f$[3] are defined with respect to the truth order $\leq_t$ and the falsity order $\leq_f$, respectively. Let us make the definition of $(\mathcal{L}_{tf}, \models_t, \models_f)$ precise. Let v be a map from the set of propositional variables into **16**. The function v is recursively extended to a function from the set of all $\mathcal{L}_{tf}$-formulas into **16** as follows:

[3] In this section we shall omit the superscripts at $\models_t^{16}$ and $\models_f^{16}$ and the subscripts at $\models_{16}^{+}$ and $\models_{16}^{-}$ when we are dealing with a 16-valued setting.

1. $v(A \wedge_t B) = v(A) \sqcap_t v(B)$;
2. $v(A \vee_t B) = v(A) \sqcup_t v(B)$;
3. $v(\sim_t A) = -_t v(A)$;
4. $v(A \wedge_f B) = v(A) \sqcup_f v(B)$;
5. $v(A \vee_f B) = v(A) \sqcap_f v(B)$;
6. $v(\sim_f A) = -_f v(A)$.

The relations $\models_t \subseteq \mathcal{P}(\mathcal{L}_{tf}) \times \mathcal{P}(\mathcal{L}_{tf})$ and $\models_f \subseteq \mathcal{P}(\mathcal{L}_{tf}) \times \mathcal{P}(\mathcal{L}_{tf})$ are defined by the following equivalences:

$$\begin{array}{lll} \Delta \models_t \Gamma & \text{iff} & \forall v \bigsqcap_t \{v(A) \mid A \in \Delta\} \leq_t \bigsqcup_t \{v(A) \mid A \in \Gamma\}; \\ \Delta \models_f \Gamma & \text{iff} & \forall v \bigsqcup_f \{v(A) \mid A \in \Gamma\} \leq_f \bigsqcap_f \{v(A) \mid A \in \Delta\}. \end{array}$$

Note that in contrast to $FOUR_2$, where $\models_t^4$ and the relation $\models_f^4$ defined in Footnote 1 are the same, in $SIXTEEN_3$ $\models_t$ and $\models_f$ are distinct relations.

Turning now to the matrix approach, we may again apply the idea of designated and antidesignated values, and in the context of $SIXTEEN_3$ it appears to be rather congenial to the definitions of $\models_t$ in terms of $\leq_t$ and $\models_f$ in terms of $\leq_f$ to consider the 16-valued logic B_{16} given with the generalized q-matrix

$$\langle \mathbf{16}, \{x \in \mathbf{16} \mid x^t \text{ is non-empty}\}, \{x \in \mathbf{16} \mid x^f \text{ is non-empty}\}, \\ \{-_t, \sqcap_t, \sqcup_t, -_f, \sqcup_f, \sqcap_f\}\rangle.$$

Moreover, for all sets of $\mathcal{L}_{tf}$-formulas Δ, Γ, semantic consequence relations $\models^+$ and $\models^-$ are defined as follows:

1. $\Delta \models^+ \Gamma$ iff for every valuation function v: (if for every $A \in \Delta$, $v(A) \in \mathcal{D}^+$, then $v(B) \in \mathcal{D}^+$ for some $B \in \Gamma$);
2. $\Delta \models^- \Gamma$ iff for every valuation function v: (if for every $A \in \Gamma$, $v(A) \in \mathcal{D}^-$, then $v(B) \in \mathcal{D}^-$ for some $B \in \Delta$).

We now have two logics, $(\mathcal{L}_{tf}, \models_t, \models_f)$ and $(\mathcal{L}_{tf}, \models^+, \models^-)$,[4] and we may note that the corresponding entailment relations of the logics $(\mathcal{L}_{tf}, \models_t, \models_f)$ and $(\mathcal{L}_{tf}, \models^+, \models^-)$ are distinct: $\models_t \neq \models^+$ and $\models_f \neq \models^-$. We may, for example, note that for every $\mathcal{L}_{tf}$-formula A, $A \models^+ \sim_f A$ and $A \models^- \sim_t A$,

[4] The latter logic is an example of what we call a *harmonious* many-valued logic, see (Wansing & Shramko, 2007).

whereas there exists no atomic $\mathcal{L}_{tf}$-formula A, such that $A \models_t \sim_f A$ or $A \models_f \sim_t A$.

Incidentally, this observation poses the following problem: Does the logic $(\mathcal{L}_{tf}, \models_t, \models_f)$ induced by the trilattice $SIXTEEN_3$ have an adequate matrix presentation? Also, a more general question arises: Under which conditions does the lattice presentation of a many-valued logic allow an equivalent matrix presentation and *vice versa*? We leave the investigation of these problems for future work.

4. Concluding remarks

We conclude this paper with an observation concerning the lattice approach towards defining a many-valued logic. Suppose that $(\mathcal{V}, \leq)$ is a complete lattice with lattice meet $\sqcap$ and lattice top $\top$. Then we may introduce an implication connective $\rightarrow$ by the following evaluation clause, compare (Hájek, 1998, p. 29):

$$v(A \rightarrow B) = \bigsqcup\{x \mid x \sqcap v(A) \leq v(B)\}.$$

The lattice operation corresponding to $\rightarrow$ is called the *residuum* of $\sqcap$ and it ensure that $\rightarrow$ satisfies the Deduction Theorem: For every valuation v, $\top \leq v(A \rightarrow B)$ iff $v(A) \leq v(B)$. If v is a valuation function, then

$$\begin{aligned} v(A) \leq v(B) \text{ iff } \top \sqcap v(A) \leq v(B) \text{ iff } \top \leq \bigsqcup\{x \mid x \sqcap v(A) \leq v(B)\} \\ \text{iff } \top \leq v(A \rightarrow B) \end{aligned}$$

(iff $\top = v(A \rightarrow B)$). It may bee seen as an advantage of the lattice approach that it admits of a general and uniform way of defining implication which is different from defining implication as material implications.

Heinrich Wansing
Institute of Philosophy
Dresden University of Technology
01062 Dresden, Germany
Heinrich.Wansing@tu-dresden.de

Yaroslav Shramko
Department of Philosophy
State Pedagogical University
500 86 Krivoi Rog, Ukraine
yshramko@ukrpost.ua

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