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Suszko's Thesis, Inferential Many-valuedness, and the Notion of a Logical System

Abstract. According to Suszko's Thesis, there are but two logical values, *true* and *false*. In this paper, R. Suszko's, G. Malinowski's, and M. Tsuji's analyses of logical two-valuedness are critically discussed. Another analysis is presented, which favors a notion of a logical system as encompassing possibly more than one consequence relation.

Keywords: Suszko's Thesis, inferential many-valuedness, many-valued logic, bivaluations, algebraic values, logical truth values.

[A] fundamental problem concerning many-valuedness is to know what it really is.
[13, p. 281]

1. Introduction

Many-valued logic is one of the oldest branches of modern formal non-classical logic. It has been developed in very influential works by Jan Łukasiewicz, Emil Post, Dmitri Bochvar, and Stephen Kleene in the 1920s and 1930s and is now a fully established and flourishing research program, see, for instance, [22] and [26]. In the 1970s, however, the prominent logician Roman Suszko called into question the theoretical foundations of many-valued logic. Suszko distinguishes between the logical values *truth* and *falsity* (*true* and *false*) on the one hand, and algebraic values on the other hand. Whereas the algebraic values are just admissible referents of formulas, the logical values play another role. One of them, *truth*, is used to define valid semantic consequence: If every premise is true, then so is (at least one of) the conclusion(s). If being false is understood as not being true, then, by contraposition, also the other logical value can be used to explain valid semantic consequence: If the (every) conclusion is not true, then so is at least one of the premises. Suszko declared that "Łukasiewicz is the chief perpetrator of a magnificent conceptual deceit lasting out in mathematical logic to the present day" [39, p. 377] and he claimed that "there are but two logical values, true and false" [9, p. 169]. This challenging claim

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is now called *Suszko's Thesis.*[1] It is has been given a formal contents by the so-called Suszko Reduction, the proof that every structural Tarskian consequence relation and hence also every structural Tarskian many-valued propositional logic is characterized by a bivalent semantics.[2]

Suszko's Reduction calls for an analysis not only because it seems to undermine many-valued logic, but also because it invites a re-consideration of the notion of a many-valued logic in particular and the notion of a logical system in general. According to Marcelo Tsuji [40, p. 308], "Suszko thought that the key to logical two-valuedness rested in the *structurality* of the abstract logics (for Wójcicki's theorem was the cornerstone of his reduction method)". Wójcicki's Theorem states that every structural Tarskian consequence relation possesses a characterizing class of matrices.[3] As pointed out in [9] and [10], Suszko's Reduction can, however, be carried out for *any* Tarskian consequence relation, and also da Costa et al. [13] emphasize that the assumption of structurality is not needed for a reduction to two-valuedness. Jean-Yves Béziau [6] noticed that a reduction to a bivalent semantics can be obtained even if a much weaker and less common notion of a logical system is assumed. In the light of this observation, Tsuji [40] criticizes Grzegorz Malinowski's analysis of Suszko's Reduction [24], [25], [27]. Malinowski takes up Suszko's distinction between algebraic many-valuedness and logical two-valuedness and highlights the bi-partition of the algebraic values into designated ones and values that are not designated, a division that plays a crucial role in the Suszko Reduction.

In fact, in the literature on many-valued logics sometimes an explicit distinction is drawn between a set $\mathcal{D}^+$ of *designated* algebraic values and a set $\mathcal{D}^-$ of *antidesignated* algebraic values, see, for example, [21], [22], [26], [31]. This distinction leaves room for values that are *neither* designated *nor* antidesignated and for values that are *both* designated *and* antidesignated, and therefore amounts to replacing logical two-valuedness (understood as a bipartition of the set of algebraic values) by logical four-valuedness in general, and by logical three-valuedness if it is postulated that $\mathcal{D}^+ \cap \mathcal{D}^- = \varnothing$ or that

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[1] Sometimes Suszko's Thesis is stated in more dramatic terms. Tsuji [40, p. 299], for example, explains that "Suszko's Thesis maintains that many-valued logics do not exist at all".

[2] Bivalent interpretations of many-valued logics have also been presented by Urquhart [41], Scott [34], and da Costa and later Béziau, see [6], [13], and references therein.

[3] Wójcicki proved his theorem for consequence operations, but the presentations in terms of consequence relations and operations are trivially interchangeable. Moreover, it should be pointed out that the entailment relation defined by a matrix in this case takes into account only valuations which are homomorphisms, cf. Section 2.

$\mathcal{D}^+ \cup \mathcal{D}^-$ is the set of all algebraic values available. The first condition is imposed by Malinowski [24], [25], [26], [27] and Gottwald [22], and the second condition may be used to define systems of paraconsistent logic.[4]

In order to provide a counterexample to Suszko's Thesis, Malinowski defined the notion of a single-conclusion *quasi*-consequence (q-consequence) relation. The semantic counterpart of q-consequence is q-entailment. Single-conclusion q-entailment is defined by requiring that if every premise is not antidesignated, then the conclusion is designated. Malinowski [24] proved that for every structural q-consequence relation, there exists a characterizing class of q-matrices, matrices which in addition to a subset of designated values $\mathcal{D}^+$ comprise a disjoint subset of antidesignated values $\mathcal{D}^-$.[5] Not every q-consequence relation has a bivalent semantics. Moreover, a q-consequence relation need not be reflexive, i.e., it may be the case that a formula is not a q-consequence of a set of formulas to which it belongs. Tsuji [40] observed that reflexivity characterizes the existence of an adequate set of bivaluations for *any* relation $\vdash \subseteq \mathcal{P}(\mathcal{L}) \times \mathcal{L}$ (where $\mathcal{P}(\mathcal{L})$ is the powerset of the propositional language $\mathcal{L}$) i.e., for any abstract logical structure in the sense of Beziau's Universal Logic [4].

In this paper, we argue that Malinowski's analysis in a sense *does* capture the central aspect of explaining the feasibility of Suszko's reduction and the distinction between algebraic and logical values. If the idea of *logical* many-valuedness as opposed to a multiplicity of algebraic values is taken seriously, then Tsuji's [40] analysis is question-begging, because it presupposes a notion of a logical system that admits at most logical two-valuedness if the single assumed consequence relation is reflexive. With another concept of a logical system, every such system is logically k-valued for some $k \in \mathbb{N}$ ($k \geq 2$), and, moreover, the definition of a logical system is such that every entailment relation in a logically k-valued logic is reflexive. By increasing the number of logical values (considered as separate subsets of the set of algebraic values) from two to three, Malinowski did a step towards logical many-valuedness, but in a way that gives up the idea of entailment as preservation of a logical value (from the premises to the conclusion, or vice versa) and thereby at the price of violating reflexivity. In the present paper, it is suggested to take a further step and to admit an increase not only of the number of logical values, but also of the number of entailment relations.

To keep this article readable and self-contained, we review Suszko's reduction method in Section 2 and there also collect some well-known facts

[4]In [43] it is explained why it is not at all unreasonable to admit $\mathcal{D}^+ \cap \mathcal{D}^- \neq \varnothing$.

[5]Also Malinowski considers only valuations which are homomorphisms.

about consequence relations. In Sections 3 and 4 we present and briefly discuss Malinowski's and Tsuji's analyses of Suszko's Thesis, and in Section 5, we present our own analysis. Finally, in Section 6, we briefly mention yet another analysis leading to higher-arity consequence relations, and in Section 7 we draw some general conclusions.

2. Suszko's reduction

Suszko suggested his reduction to a bivalent semantics with respect to the standard notion of a structural Tarskian logic (structural Tarskian consequence relation) and the standard notion of an n-valued matrix (semantically defined n-valued logic).

Let $\mathcal{L}$ be a language in a denumerable set of sentence letters and a finite non-empty set of finitary connectives $\mathcal{C} = \{c_1, \ldots, c_m\}$. A Tarskian consequence relation on $\mathcal{L}$[6] is a relation $\vdash \subseteq \mathcal{P}(\mathcal{L}) \times \mathcal{L}$ such that for every $A, B \in \mathcal{L}$ and every Δ, $\Gamma \subseteq \mathcal{L}$:

$$\Delta \cup \{A\} \vdash A \quad \text{(Reflexivity)} \tag{I}$$

$$\text{If } \Delta \vdash A \text{ then } \Delta \cup \Gamma \vdash A \quad \text{(Monotonicity)} \tag{II}$$

$$\text{If } \Delta \vdash A \text{ and } \Gamma \cup \{A\} \vdash B, \text{ then } \Delta \cup \Gamma \vdash B \quad \text{(Cut)} \tag{III}$$

A Tarskian consequence relation $\vdash$ on the language $\mathcal{L}$ is called *structural* iff for every $A \in \mathcal{L}$, every $\Delta \subseteq \mathcal{L}$, and every uniform substitution function σ on $\mathcal{L}$ (every endomorphism of the absolutely free algebra $(\mathcal{L}, c_1, \ldots, c_m)$) we have

$$\Delta \vdash A \text{ iff } \sigma(\Delta) \vdash \sigma(A) \quad \text{(Structurality)}, \tag{IV}$$

where $\sigma(\Delta) = \{\sigma(B) \mid B \in \Delta\}$. It is standard terminology to call a pair $(\mathcal{L}, \vdash)$ a Tarskian (structural Tarskian) logic iff $\vdash$ is Tarskian (structural and Tarskian).

An n-valued matrix based on $\mathcal{L}$ is a structure $\mathfrak{M} = \langle \mathcal{V}, \mathcal{D}, \{f_c : c \in \mathcal{C}\}\rangle$, where $\mathcal{V}$ is a non-empty set of cardinality $n \geq 2$, $\mathcal{D}$ is a non-empty proper subset of $\mathcal{V}$, and every f_c is a function on $\mathcal{V}$ with the same arity as c. The elements of $\mathcal{V}$ are usually called *truth values* (or truth degrees), and the elements of $\mathcal{D}$ are regarded as the *designated* truth values. In Suszko's terminology, $\mathcal{V}$ is the set of algebraic values, whereas $\mathcal{D}$ and its complement represent the two logical truth values. A valuation function v in $\mathfrak{M}$ is

[6] Often, consequence relations are defined for arbitrary sets. In this paper we restrict our interest, however, to consequence relations defined on propositional languages.

a function from $\mathcal{L}$ into $\mathcal{V}$. Also a structure $\mathfrak{M} = \langle \mathcal{V}, \mathcal{D}, \{f_c : c \in \mathcal{C}\}\rangle$ may be viewed as a logic, because the set of designated truth values determines a Tarskian (semantical) consequence relation $\models_{\mathfrak{M}} \subseteq \mathcal{P}(\mathcal{L}) \times \mathcal{L}$ by defining $\Delta \models_{\mathfrak{M}} A$ iff for every valuation v in $\mathfrak{M}$, $v(\Delta) \subseteq \mathcal{D}$ implies $v(A) \in \mathcal{D}$, where $v(\Delta) = \{v(B) \mid B \in \Delta\}$. Usually, only truth-functional valuations are considered. A valuation v in $\mathfrak{M}$ is truth-functional (structural) iff it is a homomorphism from $(\mathcal{L}, c_1, \ldots, c_m)$ into $(\mathcal{V}, f_{c_1}, \ldots, f_{c_m})$. A pair $\mathcal{M} = \langle\langle \mathcal{V}, \mathcal{D}, \{f_c : c \in \mathcal{C}\}\rangle, v\rangle$, where $\mathfrak{M} = \langle \mathcal{V}, \mathcal{D}, \{f_c : c \in \mathcal{C}\}\rangle$ is an n-valued matrix and v a valuation in $\mathfrak{M}$, may be called an n-valued model based on $\mathfrak{M}$. A model $\langle \mathfrak{M}, v\rangle$ is structural iff v is a truth-functional valuation in $\mathfrak{M}$. Obviously, an n-valued model $\mathcal{M}$ determines a Tarskian (semantical) consequence relation $\models_{\mathcal{M}} \subseteq \mathcal{P}(\mathcal{L}) \times \mathcal{L}$ by defining $\Delta \models_{\mathcal{M}} A$ iff $v(\Delta) \subseteq \mathcal{D}$ implies $v(A) \in \mathcal{D}$.

A Tarskian logic $\langle \mathcal{L}, \vdash\rangle$ is said to be characterized by an n-valued matrix $\mathfrak{M}$ iff $\vdash = \models_{\mathfrak{M}}$, $\langle \mathcal{L}, \vdash\rangle$ is characterized by an n-valued model $\mathcal{M}$ iff $\vdash = \models_{\mathcal{M}}$, $\langle \mathcal{L}, \vdash\rangle$ is characterized by a class $\mathfrak{K}$ of n-valued matrices iff $\vdash = \bigcap\{\models_{\mathfrak{M}} \mid \mathfrak{M} \in \mathfrak{K}\}$, and, finally, $\langle \mathcal{L}, \vdash\rangle$ is characterized by a class $\mathfrak{K}$ of n-valued models iff $\vdash = \bigcap\{\models_{\mathcal{M}} \mid \mathcal{M} \in \mathfrak{K}\}$. Ryszard Wójcicki [44] showed that every structural Tarskian logic is characterized by its so-called Lindenbaum bundle:

$$\{\langle\langle \mathcal{L}, \{A \in \mathcal{L} \mid \Delta \vdash A\}, \mathcal{C}\rangle, v\rangle \mid \Delta \subseteq \mathcal{L}, v \text{ is an endomorphism of } \mathcal{L}\}.$$

Theorem 2.1. (Wójcicki) *Every structural Tarskian logic is characterized by a class of structural n-valued models, for some $n \leq \aleph_0$.*

Theorem 2.2. (Suszko [39], Malinowski [26])[7] *Every structural Tarskian logic is characterized by a class of two-valued models.*

Proof. Let $\Lambda = \langle \mathcal{L}, \vdash\rangle$ be a structural Tarskian logic. Then, by Wójcicki's Theorem, Λ is characterized by a class $\mathfrak{C}_\Lambda$ of structural n-valued models. For $\langle\langle \mathcal{V}, \mathcal{D}, \{f_c : c \in \mathcal{C}\}\rangle, v\rangle \in \mathfrak{C}_\Lambda$, the function t_v from $\mathcal{L}$ into $\{0,1\}$ is defined as follows:

$$t_v(A) = \begin{cases} 1 & \text{if} \quad v(A) \in \mathcal{D} \\ 0 & \text{if} \quad v(A) \notin \mathcal{D} \end{cases}$$

The class $\{\langle\langle\{0,1\}, \{1\}, \{f_c : c \in \mathcal{C}\}\rangle, t_v\rangle \mid \langle\langle \mathcal{V}, \mathcal{D}, \{f_c : c \in \mathcal{C}\}\rangle, v\rangle \in \mathfrak{C}_\Lambda\}$ of 2-valued models characterizes Λ. ■

[7] As Caleiro et al. [10] point out, "there seems to be no paper where Suszko explicitly formulates (SR)[the Suszko Reduction] in full generality!".

Malinowski [27, p. 79] highlights (though in other words) that the Suszko Reduction does not establish the existence of a characterizing class of *structural* two-valued models. Suszko was fully aware of this fact. In [39, p. 378] he explains that

> the logical valuations are morphisms (of formulas to the zero-one model) in some exceptional cases, only. Thus, the logical valuations and the algebraic valuations are functions of quite different conceptual nature. The former relate to the truth and falsity and, the latter represent the reference assignments.

This conception may explain why the sets $\mathcal{V}$ are regarded by Suszko as sets of "algebraic values". They are algebraic, because they are assigned to formulas by valuations which are homomorphisms. If the elements of $\mathcal{V}$ need not be assigned by homomorphisms, they may be called *referential values.* The semantics which emerges as a result of Suszko's Reduction is referentially bivalent and hence logically bivalent in the sense that there exist only two distinct non-empty proper subsets of the set of algebraic values, see also Section 5.2. The fact that the valuations t_v need not be homomorphisms deprives the Suszko Reduction of much of its jeopardizing effect on many-valued logic, if structurality is viewed as a *defining* property of a logic. It has been emphasized by da Costa et al. [13] and Caleiro et al. [9], [10], however, that the condition of structurality can be given up.

Theorem 2.3. *Every Tarskian logic is characterized by a class of n-valued models, for some $n \leq \aleph_0$.*

Theorem 2.4. *Every Tarskian logic is characterized by a class of two-valued models.*

The proofs of Theorem 2.2 and 2.4 are non-constructive. Caleiro et al. [9], [11] considerably improve upon Suszko's Reduction by defining an effective method for associating to each structural finitely-valued model an equivalent two-valued model (under the assumption of effectively separable truth values).

3. Malinowski's analysis of Suszko's Thesis

Malinowski [27, p. 80 f.] succinctly analyses Suszko's reduction as follows:

> [L]ogical two-valuedness ... is obviously related to the division of the universe of interpretation into two subsets of elements: distinguished

> and others. It also turned out, under the assumption of structurality, that Tarski's concept of consequence may be considered as a "bivalent" inference operation.

He then describes his response to the question "whether logical many-valuedness is possible at all" as giving

> an affirmative answer to this question by invoking a formal framework for reasoning admitting rules of inference which lead from non-rejected assumptions to accepted conclusions.

This approach may be viewed as taking 'true' and 'false' to be predicates that give rise to contrary instead of contradictory pairs of sentence. As such, the pair 'true' versus 'false' is reflected by the contrary pairs 'designated' versus 'antidesignated' and 'accepted' versus 'rejected'. Admitting algebraic values that are neither designated nor antidesignated amounts to admitting in addition to the logical values *true* and *false* the third logical value *neither true nor false*. In other words, being false is distinguished from not being true. Whereas the algebraic values that are not designated are already given with the set of designated values $\mathcal{D}$ as its set-theoretical complement, the treatment of *true* and *false* as values that are independent of each other leads to distinguishing a non-empty set $\mathcal{D}^+$ of designated algebraic values from a non-empty set $\mathcal{D}^-$ of antidesignated algebraic values.

Let $\mathcal{L}$ be as above. An n-valued q-matrix (quasi-matrix) based on $\mathcal{L}$ is defined by Malinowski as a structure $\mathfrak{M} = \langle \mathcal{V}, \mathcal{D}^+, \mathcal{D}^-, \{f_c : c \in \mathcal{C}\}\rangle$, where $\mathcal{V}$ is a non-empty set of cardinality $n \geq 2$, $\mathcal{D}^+$ and $\mathcal{D}^-$ are distinct non-empty proper subsets of $\mathcal{V}$ such that $\mathcal{D}^+ \cap \mathcal{D}^- = \varnothing$, and every f_c is a function on $\mathcal{V}$ with the same arity as c. If it is not required that $\mathcal{D}^+ \cap \mathcal{D}^- = \varnothing$, we shall talk of generalized q-matrices. A valuation function v in $\mathfrak{M}$ is a function from $\mathcal{L}$ into the set of truth degrees $\mathcal{V}$, and Malinowski considers only valuations which are homomorphisms (from $(\mathcal{L}, c_1, \ldots, c_m)$ into $(\mathcal{V}, f_{c_1}, \ldots, f_{c_m})$).

To obtain a kind of entailment relation that does not admit of a reduction to a bivalent semantics, Malinowski defines such a relation, called q-entailment, as depending on both sets $\mathcal{D}^+$ and $\mathcal{D}^-$. A q-matrix $\mathfrak{M} = \langle \mathcal{V}, \mathcal{D}^+, \mathcal{D}^-, \{f_c : c \in \mathcal{C}\}\rangle$ determines a q-entailment relation $\models_{\mathfrak{M}} \subseteq \mathcal{P}(\mathcal{L}) \times \mathcal{L}$ by defining $\Delta \models_{\mathfrak{M}} A$ iff for every (in Malinowski's case homomorphic) valuation v in $\mathfrak{M}$, $v(\Delta) \cap \mathcal{D}^- = \varnothing$ implies $v(A) \in \mathcal{D}^+$. A pair $\mathcal{M} = \langle\langle \mathcal{V}, \mathcal{D}^+, \mathcal{D}^-, \{f_c : c \in \mathcal{C}\}\rangle, v\rangle$, where $\mathfrak{M} = \langle \mathcal{V}, \mathcal{D}^+, \mathcal{D}^-, \{f_c : c \in \mathcal{C}\}\rangle$ is an n-valued q-matrix and v a valuation in $\mathfrak{M}$, may be called an n-valued q-model based on $\mathfrak{M}$. The relation $\models_{\mathcal{M}} \subseteq \mathcal{P}(\mathcal{L}) \times \mathcal{L}$ determined by such a model is defined by the following equivalence: $\Delta \models_{\mathcal{M}} A$ iff $v(\Delta) \cap \mathcal{D}^- = \varnothing$

implies $v(A) \in \mathcal{D}^+$. A model $\langle \mathfrak{M}, v\rangle$ is structural iff v is a truth-functional valuation in $\mathfrak{M}$. If $\mathfrak{M} = \langle \mathcal{V}, \mathcal{D}^+, \mathcal{D}^-, \{f_c : c \in \mathcal{C}\}\rangle$ is a q-matrix and $\mathcal{D}^+$ is not the complement of $\mathcal{D}^-$, there is no class of functions from $\mathcal{L}$ into $\{1, 0\}$ such that $\Delta \models_{\mathfrak{M}} A$ iff for every function v from that class, $v(\Delta) \subseteq \{1\}$ implies $v(A) = 1$.[8] Let $\mathcal{M} = \langle\langle \mathcal{V}, \mathcal{D}^+, \mathcal{D}^-, \{f_c : c \in \mathcal{C}\}\rangle, v\rangle$ be an n-valued q-model. Malinowski pointed out that an equivalent three-valued q-model $\mathcal{M}' = \langle\langle\{0, \frac{1}{2}, 1\}, \{1\}, \{0\}, \{f_c : c \in \mathcal{C}\}\rangle, t_v\rangle$ can be defined as follows:

$$t_v(A) = \begin{cases} 1 & \text{if} \quad v(A) \in \mathcal{D}^+ \\ \frac{1}{2} & \text{iff} \quad v(A) \in \mathcal{V} \setminus (\mathcal{D}^+ \cup \mathcal{D}^-) \\ 0 & \text{if} \quad v(A) \in \mathcal{D}^- \end{cases}$$

A q-entailment relation $\models_{\mathfrak{M}}$ is a special case of what Malinowski calls a q-consequence relation. A q-consequence relation on $\mathcal{L}$ is a relation $\Vdash\, \subseteq \mathcal{P}(\mathcal{L}) \times \mathcal{L}$ such that for every $A \in \mathcal{L}$ and every Δ, $\Gamma \subseteq \mathcal{L}$:

$$\text{If } \Delta \Vdash A \text{ then } \Delta \cup \Gamma \Vdash A \ \ \text{(Monotonicity)} \tag{V}$$

$$\Delta \cup \{B \mid \Delta \Vdash B\} \Vdash A \text{ iff } \Delta \Vdash A \ \ \text{(Quasi-closure)} \tag{VI}$$

A q-consequence relation on $\mathcal{L}$ is called *structural* iff for every $A \in \mathcal{L}$, every $\Delta \subseteq \mathcal{L}$, and every uniform substitution function σ on $\mathcal{L}$ we have

$$\Delta \Vdash A \text{ iff } \sigma(\Delta) \Vdash \sigma(A) \ \ \text{(Structurality).} \tag{VII}$$

A pair $(\mathcal{L}, \Vdash)$ is said to be a q-logic, it is structural iff $\Vdash$ is structural. A q-logic $\langle \mathcal{L}, \Vdash\rangle$ is said to be characterized by an n-valued q matrix $\mathfrak{M}$ iff $\Vdash\, = \,\models_{\mathfrak{M}}$, $\langle \mathcal{L}, \Vdash\rangle$ is characterized by an n-valued q-model $\mathcal{M}$ iff $\Vdash\, = \,\models_{\mathcal{M}}$, $\langle \mathcal{L}, \Vdash\rangle$ is characterized by a class $\mathfrak{K}$ of n-valued q-matrices (q-models) iff $\Vdash\, = \bigcap\{\models_{\mathfrak{M}} \mid \mathfrak{M} \in \mathfrak{K}\}$ ($\Vdash\, = \bigcap\{\models_{\mathcal{M}} \mid \mathcal{M} \in \mathfrak{K}\}$).

If $\Delta \subseteq \mathcal{L}$, let $\mathcal{D}^+_\Delta = \{A \in \mathcal{L} \mid \Delta \vdash A\}$ and $\mathcal{D}^-_\Delta = \mathcal{L} \setminus (\Delta \cup \{A \in \mathcal{L} \mid \Delta \vdash A\})$. Malinowski [24] showed that every structural q-logic is characterized by the following Lindenbaum bundle:

$$\{\langle\langle \mathcal{L}, \mathcal{D}^+_\Delta, \mathcal{D}^-_\Delta, \mathcal{C}\rangle, v\rangle \mid \Delta \subseteq \mathcal{L}, v \text{ is an endomorphism of } \mathcal{L}\}.$$

[8]In [5, p. 120] it is stated that "Malinowski constructs (using an extended concept of a matrix) a consequence relation which has no two-valued logical semantics because it fails to obey the "identity" axiom of Tarski. However it has been shown (cf. [Krause/Béziau 1997]) that we can adapt in some way two-valued logical semantics even in the case of such kind of consequence relation". Note, that this 'adaptation' is sketched in terms of *two* functions mod_1, mod_2 each assigning to every formula and every set of formulas a class of models.

Theorem 3.1. (Malinowski) *Every structural q-logic is characterized by a class of structural n-valued q-models, for some $n \leq \aleph_0$.*

By the above definition of three-valued q-models $\mathcal{M}'$ and by the Suszko Reduction for the case that $\mathcal{V} \setminus (\mathcal{D}^+ \cup \mathcal{D}^-) = \varnothing$, it follows that q-logics are logically two-valued or three-valued.

Theorem 3.2. (Malinowski) *Every structural q-logic is characterized by a class of two-valued q-models or by a class of three-valued q-models.*

Another non-standard entailment relation, based on a certain generalized q-matrix, has been presented in [12] and is there said not to be "overly outlandish or inconceivable", although it fails to be a Tarskian consequence relation. The "tonk-consequence" relation of [12] for a given language $\mathcal{L}$ and set of connectives $\mathcal{C}$ is the relation $\models \subseteq \mathcal{P}(\mathcal{L}) \times \mathcal{L}$ defined for the generalized q-matrix $\mathfrak{M} = \langle \{\varnothing, \{T\}, \{F\}, \{T, F\}\}, \{\{T\}, \{F, T\}\}, \{\{F\}, \{T, F\}\}, \{f_c : c \in \mathcal{C}\} \rangle$ as follows: $\Delta \models A$ iff either for every truth-functional valuation v in $\mathfrak{M}$, $v(\Delta) \subseteq \mathcal{D}^+$ implies $v(A) \in \mathcal{D}^+$, or for every truth-functional valuation v in $\mathfrak{M}$, $v(\Delta) \cap \mathcal{D}^- = \varnothing$ implies $v(A) \notin \mathcal{D}^-$. Since tonk-entailment is not transitive, sound truth tables for Prior's connective `tonk` are available such that this addition of `tonk` does not have a trivializing effect (but see also [42]).

4. Tsuji's analysis of Suszko's Thesis

Tsuji [40, p. 305] emphasizes that Malinowski's analysis "is not wrong - but that it misses the main point of logical two-valuedness." Taking up Beziau's idea of a Universal Logic [4], Tsuji assumes a very general notion of a logical system. An *abstract logical structure* is any pair $\langle S, \vdash \rangle$, where S is an arbitrary set and $\vdash \subseteq \mathcal{P}(S) \times S$. We shall again restrict our attention to relations $\vdash \subseteq \mathcal{P}(\mathcal{L}) \times \mathcal{L}$, where $\mathcal{L}$ is a propositional language.

Theorem 4.1. (Tsuji [40]) *An abstract logical structure $\langle \mathcal{L}, \vdash \rangle$ is characterized by a class of two-valued models iff $\vdash$ satisfies (Reflexivity).*

Proof. Let $\vdash$ be reflexive and consider the class of two-valued models $\mathfrak{C} =$

$$\{\langle\langle \{0, 1\}, \{1\}, \{f_c : c \in \mathcal{C}\}\rangle, v_\Delta\rangle \mid \Delta \subseteq \mathcal{L},\, v_\Delta(A) = 1 \text{ iff } \Delta \vdash A\}.$$

Obviously, $\Delta \vdash A$ implies $\Delta \models_{\mathfrak{C}} A$. Suppose $\Delta \nvdash A$. Then $v_\Delta(A) = 0$. By (Reflexivity), $v_\Delta(\Delta) \subseteq 1$. Therefore $\Delta \not\models_{\mathfrak{C}} A$. Conversely, if $\vdash$ is not reflexive, there exist $\Delta \subseteq \mathcal{L}$ and $A \in \Delta$ with $\Delta \nvdash A$. But $\Delta \models_{\mathcal{M}} A$ for any two-valued model $\mathcal{M}$, since $A \in \Delta$. ■

Tsuji [40, p. 308] takes this result to reveal that "the problem of logical two-valuedness has more to do with the "geometrical" properties of the" relation $\vdash$ of an abstract logical structure $\langle S, \vdash \rangle$, "than with the algebraic properties of it's set S or of it's Lindenbaum bundle". In the next section we shall raise doubts about this analysis.

5. Logical n-valuedness as inferential many-valuedness

5.1. What is a logical value?

Suszko does not define the notion of a logical value except for stating that *true* and *false* are the only logical values, but he distinguishes the logical values form algebraic values and explains that the possibly many algebraic values are denotations of (propositional) formulas. Moreover, he claims that "any multiplication of logical values is a mad idea" [39, p. 378]. In any case the question arises, by virtue of which properties *true* and *false* are to be considered as logical values. The logical value *true* is given with the notion of entailment and the specification of a set of distinguished algebraic values. A formula A is entailed by a set of premises Δ if and only if it is the case that if every premise is true (alias designated), then so is the conclusion. Thus, truth is what is preserved in a valid inference from the premises to the conclusion. Let us refer to this notion of entailment as t-entailment. A formula A is logically true iff A is t-entailed by the empty set (iff for every assignment v of algebraic values to the formulas of the language under consideration, $v(A)$ is designated), and A is logically false iff A t-entails the empty set (iff for every assignment v, $v(A)$ is not designated). If we consider a bi-partition of the set of algebraic values, truth is identified with a non-empty subset $\mathcal{D}$ of designated algebraic values and falsity with the complement of $\mathcal{D}$. Now, it is characteristic for falsity to be preserved in the reverse direction, i.e., from the conclusion to at least one of the premises. One might wish to consider a notion of f-entailment understood as the backward preservation of values associated with falsity. Obviously, membership in the complement of $\mathcal{D}$ is preserved from the conclusion to the premises, but this gives *the very same* entailment relation. Since $\mathcal{D}$ is uniquely determined by its complement, and vice versa, logical two-valuedness is, in fact, reduced to logical *mono-valuedness* if there is just one entailment relation defined as truth preservation form the premises to the conclusion.

However, if we treat 'falsity' *not* as a mere abbreviation for 'non-truth' and if we correspondingly distinguish not only a set $\mathcal{D}^+$ of designated algebraic values but also a set $\mathcal{D}^-$ of antidesignated algebraic values which are not obligatorily complements of each other, then f-entailment may well be different from t-entailment.[9] Indeed, consider, e.g., the q-matrix $Ł_3{}^* := \langle\{T, \varnothing, F\}, \{T\}, \{F\}, \{f_c : c \in \{\sim, \wedge, \vee, \supset\}\}\rangle$, where the functions f_c are the usual operations of Łukasiewicz's three-valued logic (see, e.g., [28], [29]). Let us call $Ł_3{}^*$ Łukasiewicz's q-matrix. It is easy to see (cf. [43]) that in $Ł_3{}^*$ t-entailment and f-entailment do not coincide, for $A \wedge (A \supset B) \models^+ B$, but $A \wedge (A \supset B) \not\models^- B$. In this situation it would hardly be justifiable to prefer one entailment relation over the other and to deal, e.g., only with t-entailment and to completely disregard f-entailment. And if both entailment relations are *pari passu* (as the sets $\mathcal{D}^+$ and $\mathcal{D}^-$ are), why should one not conceive of logical systems with two (or perhaps even more) entailment relations?[10]

If logic is thought of as the theory of valid inferences, then a *logical value* may be seen as a value that is used to define in a canonical way an entailment relation on a set of formulas. By a canonical definition of entailment we mean a definition of entailment as a relation that (in the single conclusion case) preserves membership in a certain set of algebraic values either from the premises to the conclusion of inferences, or from the conclusion to the premises. Such a relation will be Tarskian (since preservation of a logical value from the conclusion to the premises means that if the conclusion possesses the value, then so does at least one of the premises, whereas preservation from the premises to the conclusion means that if every premise possesses the value, then so does the conclusion). Two logical values are independent of each other iff the canonically defined entailment relations associated with these values are distinct.

In addition to the term 'logical many-valuedness', Malinowski also uses the term 'inferential many-valuedness'. One might understand Malinowski's analysis as assuming (i) that *logical* values play a role in defining a single entailment (inference) relation and (ii) that this entailment relation need

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[9]Denoting t-entailment as $\models^+$ and f-entailment as $\models^-$, we obtain the following definitions: $\Delta \models^+ B$ iff for every valuation function v: $v(\Delta) \in \mathcal{D}^+$ implies $v(B) \in \mathcal{D}^+$; $\Delta \models^- B$ iff for every valuation function v: $v(B) \in \mathcal{D}^-$ implies $v(A) \in \mathcal{D}^-$, for some $A \in \Delta$.

[10]The idea of a logical system comprising two consequence relations may be detected already in [14, Chapter 6], where H.B. Curry distinguishes between deducibility and refutabiliy. Curry assumes a theory $\mathfrak{T}$ generated by axioms and a theory $\mathfrak{F}$ generated by *counteraxioms*. Moreover, it is assumed that a formula A is refutable, if a refutable formula is deducible from A. Curry then introduces a notion of negation as refutation by requiring (i) that the negation of every counteraxiom is provable and (ii) that the negation $\neg A$ of a formula A is provable, if the negation of a formula B is provable and B is deducible from A.

not be defined canonically. Truth and falsity are treated by Malinowski as logical values insofar as *both* sets of algebraic values are used to define q-entailment, which, however, is not defined canonically. If the idea of entailment as preservation of a logical value is given up, then entailment will not, in general, be a Tarskian relation. There are thus several issues involved: (i) the number of sets of designated values (as representing logical values), (ii) the relation of the algebraic values to the logical values, and (iii) the definition of entailment in terms of the logical values. As to the latter, we shall here join Jennings and Schotch [23, p. 89] in declaring that "we want inferability to preserve".

On *our* conception of Malinowski's term 'inferential many-valuedness', therefore it is distinctive of a logical value that it is used to *canonically* define an entailment relation.[11] A logic may then be said to be logically (or inferentially) k-valued if it is a language together with k canonically defined and pairwise distinct entailment relations on (the set of formulas of) this language. Each of these k entailment relations is Tarskian and hence, in particular, it is reflexive. Not only the logic of a q-entailment relation, but also classical propositional logic is obviously *not* logically two-valued in this sense, because these logics are defined by a language together with a single entailment relation.

If there in fact is more than one logical value, each determining a separate entailment relation, and if therefore it is reasonable to think of a logic as possibly containing more than just one entailment relation (or, syntactically speaking, more than just one consequence relation), our contribution to the discussion of Suszko's Thesis is this:

1. *Suszko's* notion of a many-valued logic as comprising a set of designated values $\mathcal{D}$ but not also a set of antidesignated values that may be distinct from the complement of $\mathcal{D}$ amounts to assuming logical mono-valuedness instead of logical two-valuedness, when it is assumed that a logic comes with a single entailment relation.

[11] This is actually *not* Malinowski's understanding of inferential many-valuedness. In [29] he explains that the chief feature of a q-consequence operation is that the repetition rule:

$$rep = (\{A\}, A \mid A \in \mathcal{L})$$

in general is not a rule of the operation. Moreover, q-consequence is the central notion of a purely *inferential* approach in the theory of propositional logics in the sense that "[t]he principal motivation behind the quasi-consequence ... stems from the mathematical practice which treats some auxiliary assumptions as mere hypotheses rather than axioms". The fact that the repetition rule is not unrestrictedly valid allows Malinowski [29, Section 2] to define two congruence relations on $\mathcal{L}$, inferential extensionality and inferential intensionality, which in general are independent of each other.

2. *Malinowski's* conception of logical many-valuedness is that logical values contribute to the definition of a single entailment relation, but there are reasons to let *every* logical value give rise to an entailment relation.
3. *Tsuji's* characterization of logical two-valuedness in terms of reflexivity, assuming Beziau's notion of an abstract logical structure containing a *single* consequence relation, is question begging, because with another concept of a logical system, in a logically k-valued logic every entailment relation is reflexive.

Assuming three algebraic values and understanding logical truth as receiving one of these values under any valuation, da Costa et al. [13, p. 292 f.] raise the question "Can logical truth also be multivalent?" and reply:

> It seems that *a priori* there is no good philosophical argument to reject this possibility, and this is another reason why we can reject Suszko's thesis.

Can one provide evidence to the effect that logical truth is multivalent? More generally, is there more than one logical value, each of which may be taken to determine its own entailment relation? It seems that Suszko did not raise fundamental doubts about *algebraic* many-valuedness, i.e., about assuming more than two algebraic values. One natural question then comes with the distinction between algebraic and logical values: Is *every* non-empty subset of a given set of algebraic values a logical value? We need not try to decide this question here, because for our present purposes it is enough to insist on falsity as a logical value not uniquely determined by truth. Instead of assuming with Frege that logic is the science of the most general laws of being true, for our purposes it is enough to assume that logic is the science of the most general laws of being true and the most general laws of being false.[12]

5.2. Another kind of counterexample

Before developing further the concept of a logical system with more than one entailment relation, we wish to present shortly another definition of an entailment relation, a relation which is, so to say, "essentially three-valued". The key point of Malinowski's reduction (the proof that every q-consequence

[12]Interpretations of distinguished sets of algebraic values need not appeal to truth or falsity. In a series of papers, Jennings, Schotch, and Brown have argued that paraconsistent logic can be developed as a logic that preserves a degree of incoherence from the premises to the conclusion of a valid inference, see [8], [23] and the references given there.

relation has a two-valued or a three-valued semantics) is to presuppose a tripartition of the set of algebraic values rather than a bi-partition. As already observed, Malinowski's q-entailment is not reflexive. It is nevertheless possible to define a reflexive entailment relation which is in a sense necessarily three-valued by employing Malinowski's idea that entailment should depend on both designated and antidesignated values which do not exhaust the set of all values available.

Let us return to the observation above that in Łukasiewicz's q-matrix $\text{Ł}_3{}^*$ t-entailment and f-entailment are not coincident. In this context we may also consider the intersection of these two relations, which can be called tf-entailment. Dunn [18, p. 11] introduces these three entailment relations and shows that they are all distinct. The latter point is crucial for defining a three-valued entailment relation which cannot be provided with a logically two-valued semantics *within the given set of algebraic values*. Namely, for the logic based on Łukasiewicz's q-matrix $\text{Ł}_3{}^*$ such a relation is represented by tf-entailment:[13]

$$\Delta \models^{+,-}_{\text{Ł}_3{}^*} B \quad \text{iff} \quad \forall v \text{ in } \text{Ł}_3{}^* : \quad \begin{array}{ll} (1) & v(\Delta) \subseteq \mathcal{D}^+ \Rightarrow v(B) \in \mathcal{D}^+; \\ (2) & v(B) \in \mathcal{D}^- \Rightarrow \exists A \in \Delta\, (v(A) \in \mathcal{D}^-)\,.^{14} \end{array}$$

Incidentally, $\models^{+,-}_{\text{Ł}_3{}^*}$ restricted to the so called "first-degree consequences" (statements of the form $A \models^{+,-}_{\text{Ł}_3{}^*} B$, where neither A nor B contains implications) represents the first-degree entailment fragment of Łukasiewicz's 3-valued logic (cf. [18, p. 15]). Now, it is not difficult to show that every tf-entailment relation and hence $\models^{+,-}_{\text{Ł}_3{}^*}$ is Tarskian. This means that, according to Theorem 2.4, $\models^{+,-}_{\text{Ł}_3{}^*}$ has an adequate logically two-valued semantics. It turns out, however, that it is impossible to define $\models^{+,-}_{\text{Ł}_3{}^*}$ by a two-valued valuational function defined on the carrier of $\text{Ł}_3{}^*$. Indeed, assume such a definition would be possible. Then we need to pick from the set $\{T, \varnothing, F\}$ a proper subset $\mathcal{D}$ of designated algebraic values. Moreover, $\mathcal{D}$ should contain T (because $\models^{+,-}_{\text{Ł}_3{}^*}$ is truth-preserving). There are only three such subsets: $\{T\}$, $\{T, \varnothing\}$, and $\{T, F\}$. But in the first case $\models^{+,-}_{\text{Ł}_3{}^*}$ would coincide with

[13] That is, we disregard for a moment the t-entailment and f-entailment taken separately, and concentrate solely on their intersection. Note, that it is most important here to have a value which is neither designated nor anti-designated, otherwise tf-entailment would trivially collapse into t-entailment (and f-entailment).

[14] A generalization of this definition to obtain a tf-entailment relation for *any* q-matrix $\mathfrak{M}$ based on a three-element set of algebraic values using Malinowski's three-valued valuation t_v (as defined on p. 412) is straightforward.

$\models^{+}_{\text{Ł}_3{*}}$ (which is impossible), in the second case $\models^{+,-}_{\text{Ł}_3{*}}$ would coincide with $\models^{-}_{\text{Ł}_3{*}}$ (which again, is impossible), and in the third case the intended definition for $\models^{+,-}_{\text{Ł}_3{*}}$ would be simply inadequate (making it falsity preserving from the premises to the conclusion).

5.3. An example of a natural bi-consequence logic

A natural example of a logic with two entailment (or consequence) relations (which can be called a *bi-consequence logic*) can be presented in the context of generalizing Nuel Belnap's idea of passing information about a given proposition to a computer (see [36], [37], [38], and for the seminal papers on Belnap's and Dunn's four-valued logic [1], [2], [17]). It turned out that in this context the resulting set of truth values is such that its elements can be partially ordered not only according to their truth content but also to their falsity content. These orderings are defined in such a way that in the definition of the truth (falsity) order no reference is made to the classical truth value F (*falsity*) (T (*truth*)) and the two orderings are not inverses of each other. In the simplest case, instead of the set of classical truth values $\mathbf{2} = \{T, F\}$ the set $\mathbf{16} = \mathcal{P}(\mathcal{P}(\mathbf{2}))$ is considered, giving the following generalized truth values:

1. $\mathbf{N} = \varnothing$	9. $\mathbf{FT} = \{\{F\}, \{T\}\}$
2. $\boldsymbol{N} = \{\varnothing\}$	10. $\mathbf{FB} = \{\{F\}, \{F, T\}\}$
3. $\boldsymbol{F} = \{\{F\}\}$	11. $\mathbf{TB} = \{\{T\}, \{F, T\}\}$
4. $\boldsymbol{T} = \{\{T\}\}$	12. $\mathbf{NFT} = \{\varnothing, \{F\}, \{T\}\}$
5. $\boldsymbol{B} = \{\{F, T\}\}$	13. $\mathbf{NFB} = \{\varnothing, \{F\}, \{F, T\}\}$
6. $\mathbf{NF} = \{\varnothing, \{F\}\}$	14. $\mathbf{NTB} = \{\varnothing, \{T\}, \{F, T\}\}$
7. $\mathbf{NT} = \{\varnothing, \{T\}\}$	15. $\mathbf{FTB} = \{\{F\}, \{T\}, \{F, T\}\}$
8. $\mathbf{NB} = \{\varnothing, \{F, T\}\}$	16. $\mathbf{A} = \{\varnothing, \{T\}, \{F\}, \{F, T\}\}$.

On $\mathbf{16}$ a truth order $\leq_t$ and a falsity order $\leq_f$ can be defined in a natural way.

DEFINITION 5.1. For every x, y in $\mathbf{16}$ let $x^t = \{y \in x \mid T \in y\}$, $x^{-t} = \{y \in x \mid T \notin y\}$, $x^f = \{y \in x \mid F \in y\}$, and $x^{-f} = \{y \in x \mid F \notin y\}$. Then $x \leq_t y$ iff $x^t \subseteq y^t$ and $y^{-t} \subseteq x^{-t}$ and $x \leq_f y$ iff $x^f \subseteq y^f$ and $y^{-f} \subseteq x^{-f}$.

Moreover, set-inclusion may be regarded as an information order on $\mathbf{16}$: $x \leq_i y$ iff $x \subseteq y$. We thereby obtain an algebraic structure that combines the three (complete) lattices $(\mathbf{16}, \leq_i)$, $(\mathbf{16}, \leq_t)$, and $(\mathbf{16}, \leq_f)$ into the *trilattice* $SIXTEEN_3 = (\mathbf{16}, \leq_i, \leq_t, \leq_f)$. $SIXTEEN_3$ is presented by a triple Hasse

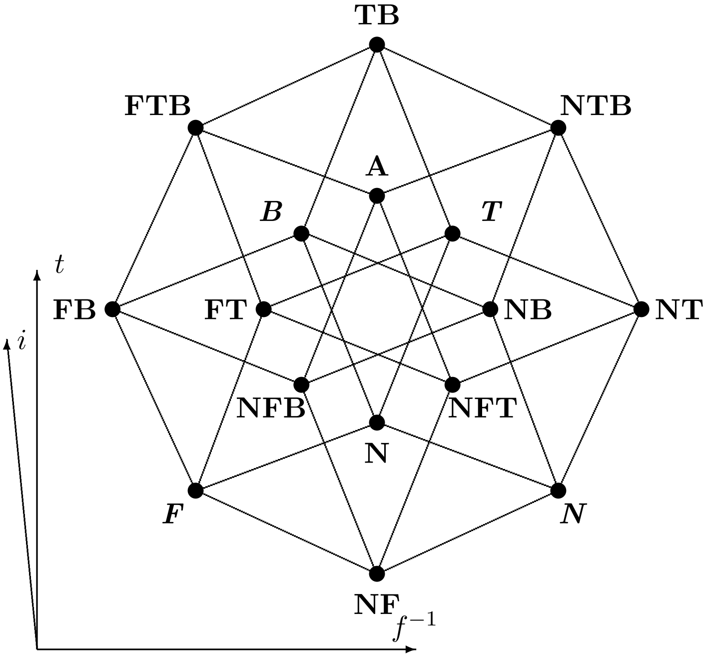

Figure 1. Trilattice $SIXTEEN_3$ (projection t – f^{-1})

diagram in Figure 1. Meets and joints exist in $SIXTEEN_3$ for all three partial orders. We will use $\sqcap$ and $\sqcup$ with the appropriate subscripts for these operations under the corresponding ordering relations. Since from the operations one can recover the relations, the trilattice $SIXTEEN_3$ may also be represented as the structure $(\mathbf{16}, \sqcap_i, \sqcup_i, \sqcap_t, \sqcup_t, \sqcap_f, \sqcup_f)$. In what follows we will be especially interested in the "logical" operations $\sqcap_t, \sqcup_t, \sqcap_f$ and $\sqcup_f$. Since the relations $\leq_t$ and $\leq_f$ are treated on a par, the operations $\sqcap_t$ and $\sqcup_t$ are not privileged as interpretations of conjunction and disjunction connectives. The operation $\sqcup_f$ may as well be regarded as a conjunction and $\sqcap_f$ as a disjunction. In other words, the logical vocabulary may be naturally split into a positive truth vocabulary and a negative falsity vocabulary. Also certain unary operations with natural negation-like properties can be defined in $SIXTEEN_3$, see Table 1. Each operation $-_\circ$ ($\circ \in \{t, f, i\}$) satisfies $-_\circ -_\circ x = x$, and $x \leq_\circ y$ implies $-_\circ y \leq_\circ -_\circ x$.

The algebraic operations $\sqcap_t$, $\sqcup_t$, $\sqcap_f$, $\sqcup_f$, $-_t$, and $-_f$ make available interpretations for logical connectives in the propositional language $\mathcal{L}_{tf}$ defined in Backus-Naur form as follows:

$$\mathcal{L}_{tf}: \quad A ::= \quad p \mid \sim_t \mid \sim_f \mid \wedge_t \mid \vee_t \mid \wedge_f \mid \vee_f$$

a	$-_t a$	$-_f a$	$-_i a$	a	$-_t a$	$-_f a$	$-_i a$
N	**N**	**N**	**A**	**NB**	**FT**	**FT**	**FT**
N	***T***	***F***	**NFT**	**FB**	**FB**	**NT**	**FB**
F	***B***	***N***	**NFB**	**TB**	**NF**	**TB**	**TB**
T	***N***	***B***	**NTB**	**NFT**	**NTB**	**NFB**	***N***
B	***F***	***T***	**FTB**	**NFB**	**FTB**	**NFT**	***F***
NF	**TB**	**NF**	**NF**	**NTB**	**NFT**	**FTB**	***T***
NT	**NT**	**FB**	**NT**	**FTB**	**NFB**	**NTB**	***B***
FT	**NB**	**NB**	**NB**	**A**	**A**	**A**	**N**

Table 1. Inversions in $SIXTEEN_3$

Definition 5.2. Let v be a map from the set of propositional variables into **16**. The function v is recursively extended to a function from $\mathcal{L}_{tf}$ into **16** as follows:

1. $v\,(A \wedge_t B) = v\,(A) \sqcap_t v\,(B)\,;$
2. $v\,(A \vee_t B) = v\,(A) \sqcup_t v\,(B)\,;$
3. $v\,(\sim_t A) = -_t v\,(A)\,;$
4. $v\,(A \wedge_f B) = v\,(A) \sqcup_f v\,(B)\,;$
5. $v\,(A \vee_f B) = v\,(A) \sqcap_f v\,(B)\,;$
6. $v\,(\sim_f A) = -_f v\,(A)\,.$

The logic of $SIXTEEN_3$ may be semantically presented as a *bi-consequence system*, viz. the structure $(\mathcal{L}_{tf}, \models_t, \models_f)$, where the two entailment relations $\models_t$ and $\models_f$ are defined with respect to the truth order $\leq_t$ and the falsity order $\leq_f$, respectively.

Definition 5.3. The set-to-set entailment relations $\models_t \subseteq \mathcal{P}(\mathcal{L}_{tf}) \times \mathcal{P}(\mathcal{L}_{tf})$ and $\models_f \subseteq \mathcal{P}(\mathcal{L}_{tf}) \times \mathcal{P}(\mathcal{L}_{tf})$ are defined by the following equivalences:

$$\Delta \models_t \Gamma \quad \text{iff} \quad \forall v \ \textstyle\bigsqcap_t \{v(A) \mid A \in \Delta\} \leq_t \bigsqcup_t \{v(A) \mid A \in \Gamma\};$$
$$\Delta \models_f \Gamma \quad \text{iff} \quad \forall v \ \textstyle\bigsqcup_f \{v(A) \mid A \in \Gamma\} \leq_f \bigsqcap_f \{v(A) \mid A \in \Delta\}.$$

What is important in the present context is that $(\mathcal{L}_{tf}, \models_t, \models_f)$ induces a bi-consequence system $(\mathcal{L}_{tf}, \models^+, \models^-)$ in the language $\mathcal{L}_{tf}$, where the entailment relations $\models^+$ and $\models^-$ are defined with respect to a certain generalized q-matrix.

Definition 5.4. The structure B_{16} is the generalized q-matrix $\langle \mathbf{16}, \{x \in \mathbf{16} \mid x^t \text{ is non-empty}\}, \{x \in \mathbf{16} \mid x^f \text{ is non-empty}\}, \{-_t, \sqcap_t, \sqcup_t, -_f, \sqcup_f, \sqcap_f\}\rangle$. That is, $\mathcal{D}^+ = \{x \in \mathbf{16} \mid x^t \text{ is non-empty}\}$, and $\mathcal{D}^- = \{x \in \mathbf{16} \mid x^f \text{ is non-empty}\}$. Moreover, for all sets of $\mathcal{L}_{tf}$-formulas Δ, Γ, semantic consequence relations $\models^+$ and $\models^-$ are canonically defined as follows:

1. $\Delta \models^{+} \Gamma$ iff for every valuation function v: (if for every $A \in \Delta$, $v(A) \in \mathcal{D}^{+}$, then $v(B) \in \mathcal{D}^{+}$ for some $B \in \Gamma$);
2. $\Delta \models^{-} \Gamma$ iff for every valuation function v: (if for every $A \in \Gamma$, $v(A) \in \mathcal{D}^{-}$, then $v(B) \in \mathcal{D}^{-}$ for some $B \in \Delta$).

Actually, B_{16} viewed as the triple $(\mathcal{L}_{tf}, \models^{+}, \models^{-})$ provides an example of what is called an *harmonious* many-valued logic in [43]. We believe that the motivation for $(\mathcal{L}_{tf}, \models_{t}, \models_{f})$, more fully presented in [36], [38], and for $(\mathcal{L}_{tf}, \models^{+}, \models^{-})$, more comprehensively presented in [43], provides sufficient motivation for assuming a generalized notion of a many-valued logic, and hence also a generalized notion of a logical system. In the next section the concept of a logically k-valued logic consisting of a language $\mathcal{L}$ together with k entailment relations on $\mathcal{L}$ is introduced.

5.4. Logically n-valued logics

In this section, we suggest to think of a (single conclusion) logic not as a *pair* $(\mathcal{L}, \vdash)$ consisting of a non-empty set of formulas $\mathcal{L}$ and a single binary derivability relation $\vdash \subseteq \mathcal{P}(\mathcal{L}) \times \mathcal{L}$, but as a $k+1$-tuple consisting of a non-empty set of formulas together with k single-conclusion binary derivability relations on $\mathcal{L}$, for some $k \geq 2$, $k \in \mathbb{N}$.

Definition 5.5. A *Tarskian k-dimensional logic* (Tarskian k-logic) is a $k+1$-tuple $\Lambda = (\mathcal{L}, \vdash_1, \ldots, \vdash_k)$ such that (i) $\mathcal{L}$ is a language in a denumerable set of sentence letters and a finite non-empty set $\mathcal{C}$ of finitary connectives, (ii) for every $i \leq k$, $\vdash_i \subseteq \mathcal{P}(\mathcal{L}) \times \mathcal{L}$, and (iii) every relation $\vdash_i$ satisfies (Reflexivity), (Monotonicity), and (Cut). Λ is said to be structural iff every $\vdash_i$ satisfies (Structurality).

Definition 5.6. Let $\mathcal{L}$ again be a language in a denumerable set of sentence letters and a finite non-empty set of finitary connectives $\mathcal{C}$. An n-valued k-dimensional matrix (k-matrix) is a structure $\mathfrak{M} = \langle \mathcal{V}, \mathcal{D}_1, \ldots, \mathcal{D}_k, \{f_c : c \in \mathcal{C}\}\rangle$, where $\mathcal{V}$ is a non-empty set of cardinality n ($2 \leq n$), $2 \leq k$, every $\mathcal{D}_i$ ($1 \leq i \leq k$) is a non-empty proper subset of $\mathcal{V}$, the sets $\mathcal{D}_i$ are pairwise distinct, and every f_c is a function on $\mathcal{V}$ with the same arity as c.[15] The sets $\mathcal{D}_i$ are called *distinguished* sets. A function from $\mathcal{L}$ into $\mathcal{V}$ is called a valuation in $\mathfrak{M}$.

[15]The notion of a k-matrix is not entirely new. Every k-matrix is a ramified matrix in the sense of Wójcicki [45, p. 189]. Ramified matrices are also called generalized matrices, see [15, p. 410 ff.]. Note, however, that Wójcicki associates with a ramified matrix *a single* entailment relation, namely $\bigcap\{\models_{\mathfrak{M}} \mid \mathfrak{M} = \langle \mathcal{V}, \mathcal{D}_i, \{f_c : c \in \mathcal{C}\}\rangle, 1 \leq i \leq k\}$.

A pair $\mathcal{M} = \langle \mathfrak{M}, v\rangle$, where $\mathfrak{M} = \langle \mathcal{V}, \mathcal{D}_1, \ldots, \mathcal{D}_k, \{f_c : c \in \mathcal{C}\}\rangle$ is an n-valued k-matrix and v a valuation in $\mathfrak{M}$, is called an n-valued k-model based on $\mathfrak{M}$. If v is a homomorphism from $(\mathcal{L}, c_1, \ldots, c_m)$ into $(\mathcal{V}, f_{c_1}, \ldots, f_{c_m})$, $\mathcal{M}$ is called a structural n-valued k-model based on $\mathfrak{M}$. Given an n-valued k-model $\mathcal{M} = \langle\langle \mathcal{V}, \mathcal{D}_1, \ldots, \mathcal{D}_k, \{f_c : c \in \mathcal{C}\}\rangle, v\rangle$, for every set $\mathcal{D}_i$ the semantic consequence relations $\models^{\Rightarrow}_{i,\mathcal{M}} \subseteq \mathcal{P}(\mathcal{L}) \times \mathcal{L}$ and $\models^{\Leftarrow}_{i,\mathcal{M}} \subseteq \mathcal{P}(\mathcal{L}) \times \mathcal{L}$ are defined as follows:

1. $\Delta \models^{\Rightarrow}_{i,\mathcal{M}} A$ iff $v(\Delta) \subseteq \mathcal{D}_i$ implies $v(A) \in \mathcal{D}_i$;
2. $\Delta \models^{\Leftarrow}_{i,\mathcal{M}} A$ iff $v(A) \in \mathcal{D}_i$ implies $v(B) \in \mathcal{D}_i$ for some $B \in \Delta$.

Obviously, the relations $\models^{\Rightarrow}_{i,\mathcal{M}}$ and $\models^{\Leftarrow}_{i,\mathcal{M}}$ are inverses of each other. We may therefore, without loss of generality, focus on the relations $\models^{\Rightarrow}_{i,\mathcal{M}}$. If we are interested in the preservation of membership in $\mathcal{D}_i$ from the conclusion to the premises, we may take the complement of $\mathcal{D}_i$ as a distinguished set of algebraic values. Moreover, the relations $\models^{\Rightarrow}_{i,\mathcal{M}}$ are all Tarskian semantic consequence relations, whence the structure $(\mathcal{L}, \models^{\Rightarrow}_{1,\mathcal{M}}, \ldots, \models^{\Rightarrow}_{k,\mathcal{M}})$ is a Tarskian k-logic.

Unsurprisingly, given an n-valued k-matrix $\mathfrak{M} = \langle \mathcal{V}, \mathcal{D}_1, \ldots, \mathcal{D}_k, \{f_c : c \in \mathcal{C}\}\rangle$, for every set $\mathcal{D}_i$ the entailment relations $\models^{\Rightarrow}_{i,\mathfrak{M}}$ and $\models^{\Leftarrow}_{i,\mathfrak{M}}$ are defined as follows:

1. $\Delta \models^{\Rightarrow}_{i,\mathfrak{M}} A$ iff for every valuation v in $\mathfrak{M}$: $v(\Delta) \subseteq \mathcal{D}_i$ implies $v(A) \in \mathcal{D}_i$;
2. $\Delta \models^{\Leftarrow}_{i,\mathfrak{M}} A$ iff for every valuation v in $\mathfrak{M}$: $v(A) \in \mathcal{D}_i$ implies $v(B) \in \mathcal{D}_i$ for some $B \in \Delta$.

A Tarskian k-logic $\langle \mathcal{L}, \vdash_1, \ldots, \vdash k\rangle$ is said to be characterized by an n-valued k-model $\mathcal{M}$ (k-matrix $\mathfrak{M}$) iff for every $\vdash_i$, $\vdash_i = \models^{\Rightarrow}_{i,\mathcal{M}}$ ($\models^{\Rightarrow}_{i,\mathfrak{M}}$). $\langle \mathcal{L}, \vdash_1, \ldots, \vdash k\rangle$ is characterized by a class $\mathfrak{K}$ of n-valued k-model (k-matrices) iff for every relation $\vdash_i$, $\vdash_i = \bigcap\{\models^{\Rightarrow}_{i,\mathcal{M}} | \langle\langle \mathcal{V}, \mathcal{D}_1, \ldots, \mathcal{D}_i, \ldots, \mathcal{D}_k, \{f_c : c \in \mathcal{C}\}\rangle, v\rangle \in \mathfrak{K}\}$ ($\vdash_i = \bigcap\{\models^{\Rightarrow}_{i,\mathfrak{M}} | \langle \mathcal{V}, \mathcal{D}_1, \ldots, \mathcal{D}_i, \ldots, \mathcal{D}_k, \{f_c : c \in \mathcal{C}\}\rangle \in \mathfrak{K}\}$).

The following statements follow immediately from Wójcicki's Theorem and Theorem 2.3.

THEOREM 5.7. *Every structural Tarskian k-logic is characterized by a class of structural n-valued k-models, for some $n \leq \aleph_0$.*

THEOREM 5.8. *Every Tarskian k-logic is characterized by a class of n-valued k-models, for some $n \leq \aleph_0$.*[16]

[16]Note that since we defined the entailment relations from the *k-models*, the semantics of *k-matrices* leaves room for other definitions of entailment.

To every relation $\vdash_i$ in a Tarskian k-logic one may apply the Suszko Reduction and the Béziau Reduction. But this observation does not raise any deep philosophical doubts about many-valued logic. Caleiro et al. [10, p. 3] are quite right in explaining that "there is some metalinguistic bivalence that one will not easily get rid off: *either* an inference obtains *or* it does not, but *not both*". If falsity is taken seriously and not just dealt with as the complement of truth, we end up, in general, with inferential four-valuedness in the form of preservation of truth, preservation of falsity, preservation of being neither true nor false, and preservation of being both true and false. For this kind of logical four-valuedness we need at least two algebraic values. Given a set $\mathcal{V}$ of algebraic values, the set of all available logical values is $\mathcal{P}(\mathcal{V}) \setminus \varnothing$. But which non-empty subsets of a given set $\mathcal{V}$ should we view as logical values? This may depend on philosophical considerations, on the intuitive interpretation of the algebraic values,[17] and on the intended applications of a logical system. In the previous section, we emphasized the naturalness of defining a truth order $\leq_t$ and a falsity order $\leq_f$ on $\mathbf{16}$, which led us to the generalized q-matrix B_{16}. In [36] it is emphasized that if we consider $\mathcal{V} = \mathbf{4} = \mathcal{P}(\mathbf{2})$ and the famous bilattice $FOUR_2$ presented in Figure 2, then truth and falsity are not dealt with as independent of each other, because in $FOUR_2$ it is assumed that T is at least as true as F. There is thus an interplay between choosing a set of algebraic values $\mathcal{V}$ and distinguished subsets $\mathcal{D}_1 \ldots, \mathcal{D}_k$ of $\mathcal{V}$.

Note also that even if one is not committed to the idea of entailment as preservation of semantical values from the premises to the conclusion(s) of an inference (or vice versa), it may still be quite natural to conceive of a logical system as comprising more than just one entailment relation. Malinowski's notion of q-entailment, for example, is in a sense one-sided. The requirement that if every premise is not antidesignated, then the conclusion is designated

[17] For Suszko, the relation between the algebraic and the logical values is established via characteristic functions, and the original intuitive interpretation of the algebraic values is uncoupled from the understanding of the logical values as *truth* and *falsity*. Similarly, the understanding of the sets of designated values $\mathcal{D}_i$ is detached from the intuitive understanding of the values in $\mathcal{V}$. It may, of course, happen that there exists a bijection between $\mathcal{V}$ and $\{\mathcal{D}_i, \ldots, \mathcal{D}_k\}$. In the context of q-consequence relations, Malinowski [27, p. 83] explains that

> for some inferentially three-valued logics based on three-element algebras, referential assignments ... and logical valuations ... do coincide. This phenomenon delineates a special class of logics, which satisfy a "generalized" version of the Fregean Axiom identifying in one-one way three logical values and three semantical correlates (or referents).

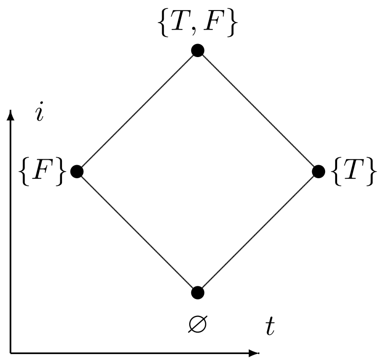

Figure 2. The bilattice $FOUR_2$

seems not to be privileged in comparison to the requirement that if every premise is designated, then the conclusion is not antidesignated. The latter notion of entailment has been studied by Frankowski [19], [20], who refers to it as p-entailment ("plausibility"-entailment). For a given interpreted language, p-entailment, need not coincide with q-entailment. Against the background of the q-matrix Ł$_3*$ with the following truth table for conjunction:

$f_\wedge$	T	$\varnothing$	F
T	T	$\varnothing$	F
$\varnothing$	$\varnothing$	$\varnothing$	F
F	F	F	F

the formula $(A \wedge B)$ p-entails A, but it is obviously not the case that $(A \wedge B)$ q-entails A. A p-consequence relation on a language $\mathcal{L}$ is a subset of $\mathcal{P}(\mathcal{L}) \times \mathcal{L}$ which is reflexive and monotonic. Frankowski shows that for every structural p-consequence relation there exists a characterizing class of q-matrices, where characterization is defined in terms of p-entailment instead of q-entailment. Interestingly, a relation which is both a q-consequence relation and a p-consequence relation is Tarskian.[18]

Mixed approaches to entailment are, of course, also possible, see [43].

[18]In [16, p. 27] it is stated (without proof) that the relations of t-entailment, f-entailment, q-entailment and p-entailment are interconnected in the following way: $\Sigma_q \subseteq \Sigma_t$, $\Sigma_q \subseteq \Sigma_f$, $\Sigma_t \subseteq \Sigma_p$, $\Sigma_f \subseteq \Sigma_p$. Here each of Σ_x stands for a set of pairs $\langle \Gamma, B \rangle$, such that Γ is a set of formulas, B is a formula, and $\Gamma \models_x B$ (where x is respectively t, f, q or p) holds in the q-matrix Ł$_3*$. Thus, it turns out that these four relations produce a lattice (relative to $\subseteq$) with Σ_q as the bottom and Σ_p as the top.

In [28], Malinowski considers an entailment relation which leads from non-accepted (alias undesignated) premises to rejected (alias false) conclusions.

6. Another analysis

Instead of increasing the number of entailment relations by associating to every set of designated values an entailment relation that preserves membership in this set from the premises to the conclusion, one may increase the number of places of the entailment relation. To every q-matrix, for instance, one might associate a ternary relation $\models \subseteq \mathcal{P}(\mathcal{L}) \times \mathcal{P}(\mathcal{L}) \times \mathcal{L}$ by postulating $\models (\Delta, \Gamma, A)$ iff for every valuation v, if $v(\Delta) \subseteq \mathcal{D}^+$ and $v(\Gamma) \subseteq \mathcal{V} \backslash (\mathcal{D}^+ \cup \mathcal{D}^-)$, then $v(A) \in \mathcal{D}^+$. Da Costa et al. [13, p. 292], e.g., explain:

> The real n-dimensional logics ($n > 2$) have to be developed by breaking down the deepest root of the principle of bivalence. We can easily imagine, for instance, a rule of deduction with *three* poles or more (our emphasis).

Higher-arity sequent systems for many-valued logics have been considered in the literature already in the 1950s by Schröter [33], and later by several other authors, see also [22] and [30]. Higher-arity sequents for modal logics have been investigated by Blamey and Humberstone [7]. The use of higher-arity sequents may have some technical and conceptual merits. However, we assume that a selection of logical values (alias non-empty subsets of some set $\mathcal{V}$) is particularly well-motivated if every logical value is associated with a dimension according to which $\mathcal{V}$ can be partially ordered (*truth*, *falsity*, *information*, *necessity*, *constructiveness*,[19] etc.). Form this point of view, it is natural to let each logical value give rise to a *binary* entailment relation.

7. Concluding remarks

In our considerations on Suszko's Thesis we argued that a logical value is a value that gives rise to an entailment relation in a canonical way. Although for each such entailment relation there exists a representation in terms of bivaluations, this does not show that it is unreasonable or unnecessary to assume more than one logical value. The observation that every entailment relation has a bivalent semantics *does*, however, cast doubt on the enterprise of providing a bivalent semantics seen as a research program that is not constrained by any further requirements, see also [3]. Indeed, as has been

[19] See, [35].

shown by Richard Routley [32] using a canonical model construction, every logic based on a λ-categorical language has a characterizing bivalent possible worlds semantics, where by a logic Routley understands an axiomatic system consisting of a countable set of axioms and a countable set of derivation rules leading from a finite number of premises to a single conclusion. Routley [32, p. 331] also notes that "if every logic on a λ-categorial language has a two-valued worlds semantics then ipso facto it has a three-valued ... semantics". This kind of 'reduction' to three-valuedness certainly does not refute the claim that there are just two truth values, and likewise the Suszko Reduction does not prove that there are but two logical values.

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