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# The Slingshot Argument and Sentential Identity

**Abstract.** The famous "slingshot argument" developed by Church, Gödel, Quine and Davidson is often considered to be a formally strict proof of the Fregean conception that all true sentences, as well as all false ones, have one and the same denotation, namely their corresponding truth value: *the true* or *the false*. In this paper we examine the analysis of the slingshot argument by means of a non-Fregean logic undertaken recently by A. Wóitowicz and put to the test her claim that the slingshot argument is in fact circular and presupposes what it intends to prove. We show that this claim is untenable. Nevertheless, the language of non-Fregean logic can serve as a useful tool for representing the slingshot argument, and several versions of the slingshot argument in non-Fregean logics are presented. In particular, a new version of the slingshot argument is presented, which can be circumvented neither by an appeal to a Russellian theory of definite descriptions nor by resorting to an analogous "Russellian" theory of $\lambda$–terms.

*Keywords*: Slingshot Argument, sentential identity, non-Fregean logic, fact ontology, situation semantics, term-forming operators, predicate abstraction.

## 1. Preliminaries: truth values and the slingshot

When Gottlob Frege introduced the notion of a truth value (see [15], [16]), he conceived it as a natural component of his language analysis, where sentences, being saturated expressions, are interpreted as a special kind of names, which refer to (denote, signify) a special kind of objects. There are, according to Frege, only two such objects: *the true* (das Wahre) and *the false* (das Falsche):

> A sentence proper is a proper name, and its Bedeutung, if it has one, is a truth-value: the True or the False ([4, p. 297]).

This revolutionary idea turned out to have a far reaching and manifold impact on the development of modern logic. It allows to complete the project of a functional analysis of language by generalizing the notion of a numerical

Presented by **Jacek Malinowski**; *Received* October 29, 2008

Special Issue: Truth Values. Part I
*Edited by* **Yaroslav Shramko and Heinrich Wansing**

*Studia Logica* (2009) 91: 429–455
DOI: 10.1007/s11225-009-9182-5

function and introducing a special kind of functions, namely propositional functions, or truth value functions, whose range of values consists of the set of truth values. (Among the most typical representatives of propositional functions one finds predicate expressions and logical connectives.) As a result one obtains a powerful tool for a conclusive implementation of the extensionality principle (also called the principle of compositionality), according to which the meaning of a complex expression is uniquely determined by the meanings of its components. On this basis one can also discriminate between extensional and intensional contexts (cf. [6]) and advance further to the conception of intensional logics.

Truth values thus prove to be an extremely effective instrument for a logical and semantical analysis of language. But is the price we pay for such an effectiveness not to high? Do we not lapse here into a typical mistake by trying to explain unclear notions like "denotation" in even more problematic terms? Perhaps truth values are just nothing but artificial *ad hoc* constructions, a sort of twilight theoretical fictions? By and large: Do sentences indeed designate the truth values?

There is a famous argument (or more precisely, a family of arguments) that is designed to provide a formally strict proof of the claim that all true sentences designate (denote, refer to) one and the same thing, as well as all false sentences do. These things are precisely the truth values: the true and the false. The argument is already anticipated (implicitly at least) by Frege (see, e.g., [16, p. 49]), and it was first formulated explicitly by Alonzo Church in his review of Carnap's *Introduction to Semantics* [7]. In *Introduction to Mathematical Logic* [8] Church reconstructs the point of his proof by means of a rather informal line of reasoning. Other remarkable versions of the argument are those by Kurt Gödel [17] and Donald Davidson [9], which make use of the formal apparatus of a theory of descriptions.

Jon Barwise and John Perry [2] have dubbed this family of arguments "the slingshot", stressing thus its extraordinary simplicity and the minimality of presuppositions involved. Versions of the slingshot argument have been analyzed in detail by many authors, see, e.g., [11], [14], [22], [23], [25], [28], [29], [32], [37], [38], and especially the most comprehensive study by Stephen Neale [24].

## 2. Reconstructing the slingshot arguments

Stated generally, the pattern of the argument goes as follows (cf. [29]). One starts with a certain sentence, and then moves, step by step, to a completely different sentence. Every two sentences in any step designate presumably one and the same thing. Hence, the starting and the concluding sentences of the argument must have the same designation as well. But the only semantically significant thing they have in common seems to be their truth value. Thus, what any sentence designates is just its truth value.

### 2.1. Church's slingshot

We first pose the argument as it is presented by Church in [8, pp. 24–25]. Note that the slingshot argument in all its forms rests essentially on the assumption that every sentence normally has a designation. Another important assumption is the principle of *substitutivity* for co-referential terms:

> If we convert a sentence into another sentence by substituting any term for a term with exactly the same designation, the resulting and the initial sentences also designate the same.

This is actually just an instance of the compositionality principle mentioned above. Now let us consider the following sequence of four sentences:

C1. Sir Walter Scott is the author of *Waverley*.

C2. Sir Walter Scott is the man who wrote 29 *Waverley* Novels altogether.

C3. The number, such that Sir Walter Scott is the man who wrote that many *Waverley* Novels altogether is 29.

C4. The number of counties in Utah is 29.

Note that this sequence does not represent a logical inference (although it is not difficult to re-articulate the argument as a formal proof by applying suitable technical machinery). It is rather a number of conversion steps each producing co-referential sentences. It is claimed that C1 and C2 have the same designation by substitutivity, for the terms "the author of *Waverley*" and "the man who wrote 29 *Waverley* Novels altogether" designate one and the same object, namely Walter Scott. And so have C3 and C4, because the number, such that Sir Walter Scott is the man who wrote that many *Waverley* Novels altogether is the same as the number of counties in Utah, namely 29. The step from C2 to C3 is justified by what Perry [29] calls *redistribution*: "rearrangement of the parts of a sentence does not effect what it designates, as long as the truth conditions remain the same". This principle may seem controversial, and Barwise and Perry [2] in fact reject it. (It must be said that they reject substitution as well.) Church himself argues that it is plausible to suppose that C2, even if not synonymous with C3, is at least so close to C3 "so as to ensure its having the same denotation". If this is indeed the case, then C1 and C4 must have the same denotation (designation). But the only (semantically relevant) thing these sentences have in common is that both are true. Thus, taking it that there must be something what the sentences designate, one concludes that it is just their truth value. As Church remarks, a parallel example involving false sentences can be constructed in the same way (by considering, e.g., "Sir Walter Scott is not the author of *Waverley*").

Church in [7] used this argument to demonstrate that sentences could not designate propositions as Carnap assumed in [5]. It seems that Carnap himself found the argument convincing enough: in his next book [6, p. 26] he *did* postulate truth values as "extensions" of sentences and, moreover, provided independent reasons for this.

### 2.2. Gödel's slingshot

Gödel in [17, pp. 128–129] highlights the impact of various theories of descriptions on the problem of what sentences designate. According to him, if—in addition to substitutivity—we take the apparently obvious view that a descriptive phrase denotes the object described, then the conclusion that "all true sentences have the same signification (as well as all false ones)" is almost inevitable. Gödel hints at a "rigorous proof" of this claim by making use of some further assumptions. Let $(\iota x)(x = a \wedge Fx)$ stand for the definite description "the $x$ such that $x$ is identical to $a$ and $x$ is $F$", and let for any sentence $X$, $[X]$ stand for what $X$ designates. Then Gödel's assumptions can be articulated as follows:

(A1) $[Fa] = [a = (\iota x)(x = a \wedge Fx)]$.

(A2) Every sentence can be transformed into an equivalent sentence of the form $Fa$. (This assumption allows Gödel to expand his argument beyond the atomic sentences, cf. [23, p. 778]).

We will next reconstruct Gödel's proof in the form of an "official" logical inference. (This reconstruction stems essentially from [23, pp. 777–779, 789], although our formulation is somewhat different). Making use of the introduced notation, we are going to prove that for any true sentences $A$ and $B$, $[A] = [B]$. To do this, we will need an additional rule of inference governing the description operator:

$$\iota\text{-INTR:} \quad \frac{A(x/a)}{a = (\iota x)(x = a \wedge A(x))}$$

where $a$ is a singular term, $A(x)$ is a sentence containing at least one free occurrence of the variable $x$, and $A(x/a)$ is the result of replacing every occurrence of $x$ in $A(x)$ by $a$.

Note the close connection between this rule and (A1). As Neale [23, p. 789] observes, $\iota$-INTR (as well as its counterpart rule $\iota$-ELIM, which goes in the opposite direction) must be a valid rule of inference in any (extensional) theory of descriptions (as it is in Russell's).

Another inference rule, which allows substitution for definite descriptions, is also taken from [23, p. 787]:

$$\iota\text{-SUB:}\quad \frac{(\iota x)\phi = (\iota x)\psi \quad A((\iota x)\phi)}{A((\iota x)\psi)} \qquad \frac{(\iota x)\phi = a \quad A((\iota x)\phi)}{A(a)} \qquad \frac{(\iota x)\phi = a \quad A(a)}{A((\iota x)\psi).}$$

With this machinery at hand, suppose the sentences G1–G3 below are true.

G1. $Fa$

G2. $a \neq b$

G3. $Gb$

Then one can proceed as follows:

G4. $a = (\iota x)(x = a \wedge Fx)$ G1, $\iota$-INTR

G5. $a = (\iota x)(x = a \wedge x \neq b)$ G2, $\iota$-INTR

G6. $b = (\iota x)(x = b \wedge Gx)$ G3, $\iota$-INTR

G7. $b = (\iota x)(x = b \wedge x \neq a)$ G2, $\iota$-INTR

G8. $(\iota x)(x = a \wedge Fx) = (\iota x)(x = a \wedge x \neq b)$ G4, G5, $\iota$-SUB

G9. $(\iota x)(x = b \wedge Gx) = (\iota x)(x = b \wedge x \neq a)$ G6, G7, $\iota$-SUB

G10. $[Fa] = [a = (\iota x)(x = a \wedge Fx)]$ (A1)

G11. $[a \neq b] = [a = (\iota x)(x = a \wedge x \neq b)]$ (A1)

G12. $[Fa] = [a = (\iota x)(x = a \wedge x \neq b)]$ G8, G10, $\iota$-SUB

G13. $[Fa] = [a \neq b]$ G11, G12, *Transitivity of* $=$

G14. $[Gb] = [b = (\iota x)(x = b \wedge Gx)]$ (A1)

G15. $[a \neq b] = [b = (\iota x)(x = b \wedge x \neq a)]$ (A1)

G16. $[Gb] = [b = (\iota x)(x = b \wedge x \neq a)]$ G9, G14, $\iota$-SUB

G17. $[Gb] = [a \neq b]$ G15, G16, *Transitivity of* $=$

G18. $[Fa] = [Gb]$ G13, G17, *Transitivity of* $=$

The inference can be easily repeated if instead of G2 we take its negation $a = b$. That is, in any case (in view of the assumption that sentences do designate) $Fa$ and $Gb$ must have one and the same designation. But $Fa$ and $Gb$ may be completely different sentences having nothing (of semantic relevance) in common, except that they are both true, as had been assumed. Then, taking into account the assumption (A2), we can easily extend this claim to any true sentences $A$ and $B$.

The paper in which Gödel put forward his argument was published in a volume from the "Library of Living Philosophers" devoted to Bertrand Russell. As is well known, Russell held that any true sentence stands for a fact. In this case the argument above would demonstrate that all true sentences stand for one and the same fact, reducing thus Russell's view to an absurdity.

Therefore, the slingshot argument sometimes has been characterized as a "collapsing argument", for what it admittedly shows is that there are fewer entities of a given kind than one might suppose ([23, p. 761]). In this sense the argument can be equally used to show that sentences do not designate situations, states of affairs or anything of the sort, leading any attempt to assume so to a breakdown of the class of supposed designata "into a class of just two entities (which might as well be called "Truth" and "Falsity")" [23, p. 761]. Another famous argument of this kind is the one by W.V. Quine (see [30], [31]), by which he intended to demonstrate that quantifying into modal contexts leads to a collapse of modality.

By the way, as Gödel in contrast to Church and Davidson offers only a rough outline of his argument, one can find in the literature several different reconstructions of Gödel's supposedly authentic proof, and it is not easy to see which one of them reproduces Gödel's line of reasoning more accurately than the others. However, such reconstructions, being different in the assumptions involved or in their technical implementation, all end up with the same conclusion. For example, if instead of (A1) one takes a more general assumption:

(A3) If $A$ and $B$ are *logically equivalent*, then $[A] = [B]$,

then one obtains another, very simple (even simplified) version of the slingshot which—though not exactly Gödelian—is clearly inspired by Gödel's ideas. Moreover, this version (see, e.g., [21]) is to some degree intermediate between Gödel's original argument and the ones developed by Church and Davidson.

Namely, let $R$ and $T$ be any true sentences and $a$ be some term which has a designation. Then we have the following four sentences which all have the same designation:

S1. $R$

S2. $a = (\iota x)(x = a \wedge R)$

S3. $a = (\iota x)(x = a \wedge T)$

S4. $T$

Indeed, S1 and S2 are logically equivalent. The same holds for S3 and S4. Now, because $(\iota x)(x = a \wedge R)$ and $(\iota x)(x = a \wedge T)$ designate one and the same object, namely $a$, by substitutivity, S2 and S3 also designate the same object. Hence, S1 and S4 must have the same designation as well. Quod erat demonstrandum.

### 2.3. Davidson's slingshot

Davidson used the slingshot to undermine the view that true sentences correspond to the facts. In [9, pp. 305–306] he explicitly enunciates assumptions needed for the argument: "that logically equivalent singular terms have the same reference; and that a singular term does not change its reference if a contained singular term is replaced by another with the same reference". Then Davidson considers any two sentences $S$ and $R$ alike in truth values and argues that the following four sentences must all have the same designation:

D1. $S$

D2. $(\iota x)(x = x \wedge S) = (\iota x)(x = x)$

D3. $(\iota x)(x = x \wedge R) = (\iota x)(x = x)$

D4. $R$

(The sentences D1, D2 as well as the sentences D3, D4 are pairwise logically equivalent and hence co-referential. And taking into account the co-referentiality of the terms $(\iota x)(x = x \wedge S)$ and $(\iota x)(x = x \wedge R)$, D2 and D3 are also co-referential.)

Thus, Davidson maintains, if we wish to claim that sentences stand for facts, we are forced to admit that all true sentences refer to one and the same fact, which Davidson [10], nimbly enough, calls *The Great Fact*. This conclusion is often employed to make a case against the correspondence theory of truth. The idea is that facts—when related to a sentence—appear non-localizable, and thus any true sentence seems to correspond to the whole universe rather than to some of its "parts". As it was suggested by C.I. Lewis [20, p. 242], a proposition refers not to some limited state of affairs, but to the "kind of *total* state of affairs we call a world". And further: "All *true* propositions have the same extension, namely, this actual world; and all *false* propositions have the same extension, namely, zero-extension." Such an understanding is eminently congenial to the Fregean account of truth values.

## 3. The slingshot argument and non-Fregean logic: Wójtowicz's challenge

In [42], Anna Wójtowicz aims at reassessing the slingshot argument by translating it into the formal language of Roman Suszko's non-Fregean logic [35] extended by the $\iota$-operator (or some suitable abstraction operator). In addition to the vocabulary of classical first-order logic (with the identity predicate), the language of non-Fregean logic contains a binary *identity connective* $\equiv$. Intuitively, a formula $A \equiv B$ states that the sentences $A$ and $B$ denote (or describe) the same situation. Note that using the notation employed in Section 2.2 this can be expressed as $A \equiv B$ iff $[A] = [B]$.

Suszko was very much animated by the idea of "abolishing" the so-called *Fregean Axiom* (see [35]) which is formulated as follows:

$$(A \leftrightarrow B) \rightarrow (A \equiv B). \qquad (FA)$$

That is, if $A$ and $B$ are materially equivalent, then they describe the same situation. Taking into account the material equivalence of all true sentences, as well as of all false sentences, this amounts to the claim that there are but two situations, one standing for all true sentences ("the true"), and another for all false ones ("the false") (see [40, p. 327]). Suszko calls the principle that "all true (and similarly all false) sentences describe the same state of affairs, that is, they have a common referent" the *semantic version of the Fregean axiom* (see [27, p. 21]).

Generally, non-Fregean logic (**NFL**) is a logic which does not validate $(FA)$. On the other hand, the slingshot argument can actually be considered as a reasoning just for the substantiation of $(FA)$, either in its syntactic or semantic form. It may even be said that the very crux of this argument is to demonstrate that the Fregean Axiom is in fact not an axiom, but can be strictly *proved* (by making a few quite natural assumptions). Therefore, the idea to employ the framework of **NFL** for an analysis of these assumptions and the argument as a whole looks quite reasonable.

**NFL** can be axiomatized in various ways. For example, one obtains the system **PCI** (predicate calculus with the identity connective) by adding to classical first-order logic with the identity predicate the following axiom schemata:

Ax1. $A \equiv B$ (where $A$ and $B$ vary in at most bound variables)

Ax2. $(A \equiv B) \to (C[A] \equiv C[A/B])$

Ax3. $(A \equiv B) \to (A \leftrightarrow B)$

Ax4. $(x_1 = y_1 \wedge \ldots \wedge x_n = y_n) \to (Px_1, \ldots, x_n \equiv Py_1, \ldots, y_n)$

Ax5. $\forall x(A \equiv B) \to (QxA \equiv QxB)$ (where $Q \in \{\forall, \exists\}$)

This formulation of **PCI** differs slightly from the one presented by Wójtowicz in [42, p. 187]. Note that the schemata $A \equiv B \to B \equiv A$ and $((A \equiv B) \wedge (B \equiv C)) \to (A \equiv C)$ are provable with the help of Ax2 (see [40, p. 325]).

Ryszard Wójcicki has shown that the only identities in **PCI** are so-called "trivial identities", i.e., expressions of the form $A \equiv B$, where $A$ and $B$ differ at most with respect to the form of bound variables. Therefore Wójtowicz finds **PCI** to be too weak to serve as a framework for an analysis of the slingshot argument. For example, **PCI** does not validate the assumption (A3) above that is needed for the version of the argument which she puts to the proof. Hence her basic system is **WBQ**[1] defined as follows:

$$\mathbf{WBQ} = \mathbf{PCI} \cup \{A \equiv B : A \leftrightarrow B \text{ is a theorem of classical predicate logic}\}.$$

In Footnote 5 of [42], Wójtowicz points out that the formulas $A$ and $B$ of the additional identities $A \equiv B$ added to **PCI** are $\equiv$–free. That is, **WBQ** is obtained by adding to **PCI** the following inference rule (CL) for all $\equiv$–free formulas $A$ and $B$:

$$\frac{\vdash A \leftrightarrow B}{\vdash A \equiv B}$$

Note that $(FA)$ is still not a theorem of **WBQ**. Clearly, (A3) is expressible in **WBQ** and in fact is provable in it. Ax4 corresponds to the principle of substitutivity of co-denoting terms. However, these principles are not enough to prove the Fregean Axiom.

As a paradigmatic version of the slingshot argument Wójtowicz takes the "intermediate" (Gödel–Davidson) version S1–S4 above. It involves the following basic thesis concerning the definite description operator:

[1] As Omyła [26, p. 13] explains, "W" stands for Wittgenstein, "B" for Boolean Algebra, and "Q" for quantifiers.

(i) $A$ is logically equivalent to $a = (\iota x)(x = a \wedge A)$.[2]

Thus, Wójtowicz adds the $\iota$-operator to the language of **WBQ**, and, to make (i) true, accepts the following axiom scheme, governing this operator:

$$(*)\ \ (\iota x)(x = a \wedge A) = a \leftrightarrow A \wedge (\iota x)(x = a \wedge A) = b \leftrightarrow {\sim}A.^{3}$$

Let $\mathbf{L}$+$(*)$ stand for a non-Fregean logic $\mathbf{L}$ with the axiom scheme $(*)$, which is not weaker than **WBQ**. Wójtowicz interprets the truth of a sentence as its belonging to a certain complete theory. Taking into account that she also assumes that a sentences is false iff its negation belongs to the theory (as it is clear from the proof of the theorem below), the theories in question must be not just complete but *negation-complete*. That is for any sentence $A$, either $A$ or ${\sim}A$ must belong to the theory. Let for a given logic $\mathbf{L}$, COMP($\mathbf{L}$) stand for the class of all negation-complete theories formulated in the language of $\mathbf{L}$. Consider the following condition:

$$(**)\ \ \forall\, \mathbf{T} \in \mathrm{COMP}(\mathbf{L}{+}(*))\ \forall A\ \forall B\ (A \in \mathbf{T} \text{ and } B \in \mathbf{T} \Rightarrow A \equiv B \in \mathbf{T}).$$

Intuitively this condition tells us that any two sentences from an arbitrary negation-complete theory have the same denotation (denote the same situation). Wójtowicz proves then the following theorem:

THEOREM 1. *For any logic* $\mathbf{L}$+$(*)$ *which is not weaker than* **WBQ**, *the condition* $(**)$ *holds if and only if $(FA)$ is provable in* $\mathbf{L}$+$(*)$.

PROOF. $\Rightarrow$: Let condition $(**)$ hold, and assume $(FA)$ is not a theorem of $\mathbf{L}$+$(*)$. That is, there exists a theory $\mathbf{T} \in \mathrm{COMP}(\mathbf{L}{+}(*))$, such that $A \leftrightarrow B \in \mathbf{T}$ and $A \equiv B \notin \mathbf{T}$. If $A \leftrightarrow B \in \mathbf{T}$, this means that either $A$ and $B$ are both true or they are both false. In both cases we easily derive a contradiction by using $(**)$. $\Leftarrow$: Obvious. ∎

[2] We simplify here Wójtowicz's exposition. Namely, she seems also to take as an *independent assumption* the thesis that for any true sentences $A$ and $B$: $a = (\iota x)(x = a \wedge A) = (\iota x)(x = a \wedge B)$. Observe, however, that it is enough to postulate that for any true sentence $A$, $a = (\iota x)(x = a \wedge A)$. On an object-language level this can be expressed either by the formula $A \rightarrow (a = (\iota x)(x = a \wedge A))$, or by the rule of inference $A \vdash (a = (\iota x)(x = a \wedge A))$. But both are derivable if (i) is given.

[3] Wójtowicz considers also another scheme, which looks as follows: $(\eta(A) = a \leftrightarrow A) \wedge (\eta(A) = b \leftrightarrow {\sim}A)$, where $\eta$ is supposed to be a suitable description (or abstraction) operator. However, if $\eta$ is instantiated by the $\iota$-operator, satisfaction of the uniqueness condition is not guaranteed. Indeed, let $A$ be true, and $a \neq b$, for some individual constants $a$ and $b$. Then, by the first conjunct of Wójtowicz's scheme, $(\iota x)A = a \leftrightarrow A$, but also $(\iota x)A = b \leftrightarrow A$. Hence $(\iota x)A = a$ and $(\iota x)A = b$, and then $a = b$, a contradiction.

It is noteworthy that the proof of Theorem 1 makes no use of condition $(*)$. Further, the following key theorem puts the slingshot argument into the context of **NFL**:

THEOREM 2. $(FA)$ *is provable in any non-Fregean logic* $\mathbf{L}+(*)$, *which is not weaker than* **WBQ**.[4]

PROOF. To prove this theorem, Wójtowicz makes use of Theorem 1. She considers an arbitrary theory $\mathbf{T} \in \mathrm{COMP}(\mathbf{L}+(*))$ and takes two arbitrary sentences $A$ and $B$ belonging to $\mathbf{T}$.

Next she observes that the following identities are theorems of $\mathbf{T}$:

1. $A \equiv a = (\iota x)(A \wedge (x = a))$
2. $a = (\iota x)(A \wedge (x = a)) \equiv a = (\iota x)(B \wedge (x = a))$
3. $B \equiv a = (\iota x)(B \wedge (x = a))$

By transitivity of the identity connective we immediately get $A \equiv B$, and thus, by Theorem 1, the required result. ■

And finally, we obtain the following corollary:

COROLLARY 3. $(*) \rightarrow (FA)$ *is a theorem of* **WBQ**.[5]

This corollary supposedly shows that it is precisely the principle $(*)$ which is responsible for the validation of the Fregean Axiom. Moreover, Wójtowicz interprets $(*)$ as attributing to $\iota$ the property of being an operator such that it "sends" all the true sentence to some (fixed) object $a$ and all the false sentences to another object $b$. This makes her to believe that $(FA)$ is "implicit in the definition of this operator" [42, p. 190], and thus, $(*)$ is the Fregean Axiom "in sheep's clothing". This might be taken, Wójtowicz concludes, as evidence for the *circularity* of the slingshot argument.

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[4] Clearly, in $\mathbf{L}+(*)$ the rule CL admits the derivation of $A \equiv B$ from $A \leftrightarrow B$ not only for classically valid $\equiv$–free biconditionals $A \leftrightarrow B$, but also for $\equiv$–free instances of the biconditionals from $(*)$.

[5] Of course, as such this formulation is not quite accurate. Remember that the language of "pure" **WBQ** does not contain the $\iota$-operator, and hence, $(*)$ is not expressible in it. Thus, what the corollary actually means is that $(*) \rightarrow (FA)$ is provable in the system $\mathbf{WBQ}^{\iota}$, which is **WBQ** *conservatively extended* with the $\iota$-operator. That is, $\iota$ is added to the language of **WBQ**, but no further principles for $\iota$ are assumed, in particular $(*)$ is not supposed.

## 4. Circulus vitiosus?

The conclusion made by Wójtowicz looks quite puzzling. Indeed, what does it mean for an argument to be circular? It should mean evidently that the conclusion of the argument is (explicitly or implicitly) assumed already[6]. To bring this to light one has to demonstrate not that the conclusion follows from the assumptions (it goes without saying), but rather that the conclusion in fact *implies* one of the assumptions. In the present case one would need to show that $(FA) \to (*)$ is a theorem of $\mathbf{WBQ}^{\iota}$ ($\mathbf{WBQ}$ conservatively extended by the $\iota$-operator), since it is exactly $(FA)$ that is the central conclusion of the slingshot argument. However, this circularity is not given as the following theorem reveals:

THEOREM 4. $(FA) \to (*)$ *is not a theorem of* $\mathbf{WBQ}^{\iota}$.

PROOF. It is enough to demonstrate that by adding $(FA)$ to $\mathbf{WBQ}^{\iota}$, $(*)$ remains unprovable. And this is indeed the case, for $\mathbf{WBQ} + (FA)$ gives us classical logic, and $(*)$ is not provable even in certain non-conservative extensions of classical logic, for example within Russell's theory of descriptions. All the more so it is not provable in classical logic to which the $\iota$-operator is added conservatively. (Note that $\mathbf{WBQ}^{\iota} + (FA) = \mathbf{WBQ} + (FA)^{\iota}$.) ■

Now, in view of this theorem and of the fact that Corollary 3 fails to establish circularity, how could Wójtowicz's claim about the circularity of the slingshot argument reasonably be understood? What she apparently means is a sort of an "informal circle" arising from the underlying informal interpretation of $(*)$ as introducing "an operator which ascribes certain arbitrarily chosen objects to all true (resp.—false) sentences" [42, p. 190]. This principle, being formulated as it is, gives indeed some ground for such an interpretation. The ground is seated in the conjunctive form of $(*)$, and especially in its second conjunct which correlates any false sentence to a $\iota$-expression denoting some fixed object. But is such a conjunctive formulation really necessary for the description operator? We believe it is not.

If we have in mind some kind of referential definite descriptions, the following principle looks much more natural:

($\star$) $(\iota x)(A(x) \wedge x = a) = a \leftrightarrow A(x/a)$, where $A(x)$ is a formula perhaps containing free occurrences of $x$, and $A(x/a)$ is the result of replacing every occurrence of $x$ in $A(x)$ by $a$.

[6] As Wójtowicz puts it: "The *circulus vitiosus* in a reasoning appears when one of the premises ... is identical—or equivalent in an obvious way—to the corollary" [42, p. 190].

Notice that if $A(x)$ contains no free occurrences of the variable $x$, then $A(x)$ (as well as $A(x/a)$) is just $A$, and $(\star)$ "degenerates" into $(\iota x)(A \wedge x = a) = a \leftrightarrow A$, which turns out to be exactly the first conjunct of $(*)$. In other words, the first conjunct of $(*)$ is just a particular case of $(\star)$.

One might wonder whether the condition attached to $(\star)$ can (and ought to) be strengthened by stipulating that $A(x)$ must *actually* contain *at least one* free occurrence of $x$ (cf. the formulation of the $\iota$-INTR rule above). However, as Dunn has remarked in [12, p. 351–352] (concerning the $\lambda$-abstractor), such a restriction in effect would be an "empty gesture", because, e.g., of the following equivalence:

$$A \leftrightarrow A \wedge (A \vee B(x)).$$

This equivalence allows one to "dummy in" occurrences of variables and to derive easily the "degenerate case" as follows: $(\iota x)(A \wedge x = a) = a \leftrightarrow (\iota x)((A \wedge (A \vee B(x))) \wedge x = a) = a \leftrightarrow (A \wedge (A \vee B(a))) \leftrightarrow A$. But notice that it would be impossible to derive $(\star)$ in full generality from its degenerate version taken as a primitive.

Next, when comparing $(*)$ with $(\star)$, one may doubt whether the second half of $(*)$, and thus $(*)$ as a whole, is really needed for the slingshot argument. To check this, let us take a look at the following modification of Theorem 2:

THEOREM 5. $(FA)$ *is provable in any non-Fregean logic* $\mathbf{L}+(\star)$ *which is not weaker than* **WBQ**.

PROOF. We first show that $(A \wedge B) \to (A \equiv B)$ is provable in $\mathbf{L}+(\star)$. We use the Deduction Theorem:

1. $A \wedge B$ (*Assumption*)
2. $A$ ($\wedge$-*Elimination*)
3. $B$ ($\wedge$-*Elimination*)
4. $A \leftrightarrow a = (\iota x)(A \wedge (x = a))$ $(\star)$
5. $B \leftrightarrow a = (\iota x)(B \wedge (x = a))$ $(\star)$
6. $a = (\iota x)(A \wedge (x = a))$ (2, 4, *modus ponens (MP)*)
7. $a = (\iota x)(B \wedge (x = a))$ (3, 5, *MP*)
8. $(\iota x)(A \wedge (x = a)) = (\iota x)(B \wedge (x = a))$ (6, 7, *Transitivity of* $=$)
9. $(\iota x)(A \wedge (x = a)) = (\iota x)(B \wedge (x = a)) \to a = (\iota x)(A \wedge (x = a)) \equiv a = (\iota x)(B \wedge (x = a))$ (Ax4)
10. $a = (\iota x)(A \wedge (x = a)) \equiv a = (\iota x)(B \wedge (x = a))$ (8, 9, *MP*)

11. $A \equiv a = (\iota x)(A \wedge (x = a))$ (4, *CL*)
12. $A \equiv a = (\iota x)(B \wedge (x = a))$ (10, 11, *Transitivity of* $\equiv$)
13. $B \equiv a = (\iota x)(B \wedge (x = a))$ (5, *CL*)
14. $A \equiv B$ (12, 13, *Transitivity of* $\equiv$)

Next, we show that $({\sim}A \wedge {\sim}B) \rightarrow (A \equiv B)$ is also provable in $\mathbf{L}+(\star)$:

1. ${\sim}A \wedge {\sim}B$ (*Assumption*)
2. ${\sim}A$ (1, $\wedge$-*Elimination*)
3. ${\sim}B$ (1, $\wedge$-*Elimination*)
4. ${\sim}A \leftrightarrow a = (\iota x)({\sim}A \wedge (x = a))$ $(\star)$
5. ${\sim}B \leftrightarrow a = (\iota x)({\sim}B \wedge (x = a))$ $(\star)$
6. $a = (\iota x)({\sim}A \wedge (x = a))$ (2, 4, *MP*)
7. $a = (\iota x)({\sim}B \wedge (x = a))$ (3, 5, *MP*)
8. $(\iota x)({\sim}A \wedge (x = a)) = (\iota x)({\sim}B \wedge (x = a))$ (8, 9, *Transitivity of* $=$)
9. ${\sim}A \equiv a = (\iota x)({\sim}A \wedge (x = a))$ (4, *CL*)
10. ${\sim}B \equiv a = (\iota x)({\sim}B \wedge (x = a))$ (5, *CL*)
11. $A \equiv {\sim}(a = (\iota x)({\sim}A \wedge (x = a)))$ (9, Ax2)
12. $B \equiv {\sim}(a = (\iota x)({\sim}B \wedge (x = a)))$ (10, Ax2)
13. $(\iota x)({\sim}A \wedge (x = a)) = (\iota x)({\sim}B \wedge (x = a)) \rightarrow {\sim}(a = (\iota x)({\sim}A \wedge (x = a))) \equiv {\sim}(a = (\iota x)({\sim}B \wedge (x = a)))$ (Ax4)
14. ${\sim}(a = (\iota x)({\sim}A \wedge (x = a))) \equiv {\sim}(a = (\iota x)({\sim}B \wedge (x = a)))$ (8, 13, *MP*)
15. $A \equiv B$ (11, 12, 14, *Transitivity of* $\equiv$)

Thus, we have both $\vdash (A \wedge B) \rightarrow (A \equiv B)$ and $\vdash ({\sim}A \wedge {\sim}B) \rightarrow (A \equiv B)$. Hence, $\vdash (A \wedge B) \vee ({\sim}A \wedge {\sim}B) \rightarrow (A \equiv B)$, and consequently, $\vdash (A \leftrightarrow B) \rightarrow (A \equiv B)$. ■

This proof shows that it is sufficient to have $(\star)$ for incorporating the slingshot argument into the context of a non-Fregean logic. Hence, Wójtowicz's $(*)$ is not necessary, in particular the second conjunct of this principle is superfluous. And thus the interpretation of $\iota$ as an operator that by definition or already by virtue of its meaning ascribes to all true (and all false) sentences one and the same referent remains unjustified. With discarding $(*)$ and rejecting the accompanying informal interpretation also the flavor of "informal circularity" connected to $(*)$ disappears.

We obtain the following corollary:

Corollary 6. $(\star) \to (FA)$ *is a theorem of* $\mathbf{WBQ}^\iota$.

This corollary expresses exactly the idea already articulated by Gödel in [17] (see Section 2.2), namely that accepting a certain kind of definite descriptions (together with some other assumptions) is sufficient to get the Fregean conception of "the true" and "the false" as the only possible denotations for sentences. As we will see in the next section, the condition imposed on such a description operator allows further weakening.

## 5. Non-Fregean logic and definite descriptions

We have shown that Wójtowicz's claim about the circularity of the slingshot argument is ill-founded. Nevertheless, as we believe, the very idea to employ non-Fregean logic for the analysis of the argument has merit by itself. Apart from a direct correlation between $(FA)$ and the conclusion of the slingshot, the technical apparatus of **NFL** provides a natural and efficient framework for formalizing and investigating its various versions. For example, it allows to adequately represent on an object language level the genuine assumption (A1) made by Gödel. Indeed, Gödel's version of the slingshot does not presuppose the co-referentiality of logically equivalent sentences, and correspondingly, Gödel need not postulate, e.g., the *logical* equivalence of "Socrates is wise" and "Socrates is the object which is wise and is identical with Socrates". Gödel merely admits that these expressions "mean the same thing" [17, p. 129], what Neal interprets in the sense that they "stand for the same fact" [23, p. 777]. As Suszko would presumably say, they denote (or describe) the same situation.

The language of a non-Fregean logic makes it possible to directly express the idea behind Gödel's assumption (A1). Let us define **PCI**+$(\bullet)$ as the system obtained from the system **PCI** (see Section 3) by extending its language with the $\iota$-operator which is subject to the following scheme:

- $(\bullet)$ $(\iota x)(A(x) \wedge x = a) = a \equiv A(x/a)$, where $A(x)$ is a formula perhaps containing free occurrences of $x$, and $A(x/a)$ is the result of replacing every occurrence of $x$ in $A(x)$ by $a$.

This scheme, being very similar to $(\star)$, does not represent the logical equivalence of two expressions, but only the identity of the corresponding situations. Moreover, $(\bullet)$ generalizes the Gödelian assumption (A1) in a twofold respect: first, it deals with arbitrary formulas (maybe containing $x$ and $a$), and not only with atomic sentences, and second, it allows $x$ and $a$ not to occur in $A$ at all.

Notice that $(\bullet)$ is derivable in **WBQ**+$(\star)$ by applying the *CL*-rule to $\star$. Yet, the system **WBQ** might be considered problematic from a rigorous "non-Fregean standpoint", since its characteristic rule *CL* comes pretty near to the Fregean axiom. So, is it maybe not the scheme for the $\iota$-operator alone, but also the rule *CL*, that bears responsibility for the validity of the slingshot argument? Some researchers (see, e.g. [2], [34]) indeed reject the view that logically equivalent sentences must have the same denotation and hope to reject in this way the slingshot argument as well.

It is therefore worth noting that $(\bullet)$ allows to represent the slingshot argument even on the basis of the weakest non-Fregean logic **PCI**.

THEOREM 7. *$(FA)$ is provable in* **PCI**+$(\bullet)$.

PROOF. *Mutatis mutandis* as in Theorem 5. ■

Note that $(\bullet)$ is the only kind of non-trivial identities within **PCI**+$(\bullet)$. At the same time $(\bullet)$ represents the minimal principle for any theory of descriptions in which definite descriptions are supposed to be singular terms that denote objects possessing some characteristics. Thus, Theorem 7 tells us that extending even the weakest non-Fregean logic with the minimal theory of descriptions of a certain kind immediately brings this logic to the collapse. Such is the destructive effect of the slingshot!

It also turns out that the context of the underlying non-Fregean logic **PCI** allows it to take seriously the restriction to an actual free occurrence of the variable $x$ in the given formula. Let us consider the following principle, which is a weakening of $(\bullet)$:

$(\bullet')$ $(\iota x)(A(x) \wedge x = a) = a \equiv A(x/a)$, where $A(x)$ is a formula containing at least one free occurrence of $x$, and $A(x/a)$ is the result of replacing every occurrence of $x$ in $A(x)$ by $a$,

and let us define the corresponding system **PCI**+$(\bullet')$. Identities like $A \equiv A \wedge (A \vee B(x))$ are not theorems of **PCI**, hence in **PCI**+$(\bullet')$ it is impossible to introduce at will "dummy" free occurrences of variables. Nevertheless, the slingshot argument (in its Gödelian version) remains in force even in this system. Namely, consider the following variation of the Fregean axiom:

$$(A(a) \leftrightarrow B(b)) \rightarrow (A(a) \equiv B(b)). \qquad (FA')$$

In $(FA')$ we are still dealing with arbitrary sentences $A(a)$ and $B(b)$ with the mere restriction that these sentences must contain arbitrary constants $a$ and $b$, respectively. Then the following theorem holds:

THEOREM 8. *$(FA')$ is provable in* **PCI**$+(\bullet')$.

PROOF. First, consider the following proof:

1. $A(a) \wedge B(b)$ $(Assumption)$
2. $a \neq b$ $(Assumption)$
3. $A(a)$ (1, $\wedge$-*Elimination*)
4. $B(b)$ (1, $\wedge$-*Elimination*)
5. $A(a) \equiv a = (\iota x)(A(x) \wedge x = a)$ $(\bullet')$
6. $B(b) \equiv b = (\iota x)(B(x) \wedge x = b)$ $(\bullet')$
7. $a \neq b \equiv a = (\iota x)(x \neq b \wedge x = a)$ $(\bullet')$
8. $a \neq b \equiv b = (\iota x)(a \neq x \wedge x = b)$ $(\bullet')$
9. $A(a) \rightarrow a = (\iota x)(A(x) \wedge x = a)$ (5, Ax3)
10. $B(b) \rightarrow b = (\iota x)(B(x) \wedge x = b)$ (6, Ax3)
11. $a \neq b \rightarrow a = (\iota x)(x \neq b \wedge x = a)$ (7, Ax3)
12. $a \neq b \rightarrow b = (\iota x)(a \neq x \wedge x = b)$ (8, Ax3)
13. $a = (\iota x)(A(x) \wedge x = a)$ (3, 9, *MP*)
14. $b = (\iota x)(B(x) \wedge x = b)$ (4, 10, *MP*)
15. $a = (\iota x)(x \neq b \wedge x = a)$ (2, 11, *MP*)
16. $b = (\iota x)(a \neq x \wedge x = b)$ (2, 12, *MP*)
17. $(\iota x)(A(x) \wedge x = a) = (\iota x)(x \neq b \wedge x = a)$ (13, 15, *Transitivity of* $=$)
18. $(\iota x)(B(x) \wedge x = b) = (\iota x)(a \neq x \wedge x = b)$ (14, 16, *Transitivity of* $=$)
19. $A(a) \equiv a = (\iota x)(x \neq b \wedge x = a)$ (5, 17, Ax4)
20. $A(a) \equiv a \neq b$ (7, 19, *Transitivity of* $\equiv$)
21. $B(b) \equiv b = (\iota x)(a \neq x \wedge x = b)$ (6, 18, Ax4)
22. $B(b) \equiv a \neq b$ (8, 21, *Transitivity of* $\equiv$)
23. $A(a) \equiv B(b)$ (20, 22, *Transitivity of* $\equiv$)

Clearly, if in this proof instead of assumption 2 we take its negation ($a = b$), we get the same conclusion. Thus, this assumption can be eliminated, and by the Deduction Theorem we get $\vdash A(a) \wedge B(b) \rightarrow A(a) \equiv B(b)$.

The proof can be repeated by taking $\sim A(x), \sim B(x), \sim A(a), \sim B(b)$ instead of $A(x), B(x), A(a), B(b)$, and thus, by the Deduction Theorem, we have $\vdash (\sim A(a) \wedge \sim B(b)) \rightarrow (\sim A(a) \equiv \sim B(b))$. By Ax2, $\vdash (\sim A(a) \equiv \sim B(b)) \leftrightarrow (A(a) \equiv B(b))$, and hence, $\vdash (\sim A(a) \wedge \sim B(b)) \rightarrow (A(a) \equiv B(b))$.

Consequently, we obtain $\vdash (A(a) \wedge B(b)) \vee (\sim A(a) \wedge \sim B(b)) \rightarrow (A(a) \equiv B(b))$, and then, $\vdash (A(a) \leftrightarrow B(a)) \rightarrow (A(a) \equiv B(b))$. ∎

Suszko conceived his non-Fregean logic as a foundation for a theory of situations in a Wittgensteinean spirit. Since **PCI** is usually regarded the weakest (or the basic) system of non-Fregean (predicate) logic, Theorems 7 and 8 indicate that a theory of situations based on Suszko's non-Fregean logic is in fact incompatible with any theory of descriptions in which "Sir Walter Scott is the man that wrote *Waverley* and is identical with Walter Scott" means the same (and hence describes the same situation) as "Sir Walter Scott wrote *Waverley*".

## 6. Non-Fregean logic and $\lambda$–expressions

In this section we examine the effect of combining Suszko's non-Fregean logic with the $\lambda$–abstractor $(\lambda x)$. Indeed, it is possible to prove the analogues of Theorems 7 and 8 by using $\lambda$–terms instead of $\iota$–terms. We adopt a standard interpretation of a *lambda-term* $\lambda x A(x)$ as "the class of all $x$ such that $A(x)$" [7, p. 299]. Then we can make some natural assumption (analogous to the corresponding principle taken for the $\iota$–operator) about $\lambda$–terms and sentential, descriptive identity, which uses the $\lambda$–operator twice. Again, it is possible to consider two versions of this assumption:

($\circ$) $(\lambda x(A(x) \wedge x = a) = \lambda x(x = a)) \equiv A(x/a)$, where $A(x)$ is a formula *perhaps* containing free occurrences of $x$, and $A(x/a)$ is the result of replacing every occurrence of $x$ in $A(x)$ by $a$.

($\circ'$) $(\lambda x(A(x) \wedge x = a) = \lambda x(x = a)) \equiv A(x/a)$, where $A(x)$ is a formula containing *at least one* free occurrence of $x$, and $A(x/a)$ is the result of replacing every occurrence of $x$ in $A(x)$ by $a$.

We now consider the systems **PCI**+($\circ$) and **PCI**+($\circ'$) in the first-order language with the $\lambda$–operator. Then evidently the following theorems hold true:

THEOREM 9. $(FA)$ *is provable in* **PCI**+($\circ$).

THEOREM 10. $(FA')$ *is provable in* **PCI**+($\circ'$).

The proofs of these theorems are completely analogous to the proofs of Theorems 7 and 8, respectively. Note that no specific assumptions about equality between $\lambda$–terms from the theory of $\lambda$–conversion are used in this formalized ($\lambda$-)versions of the slingshot argument.

Let us next add to the language of **PCI** the new predicate constant $\in$ and consider expressions of the form $a \in \lambda x A(x)$. This is fully in accord with the set-theoretical (extensional) understanding of the lambda terms accepted above. The expression $a \in \lambda x A(x)$ is then a formula, and one can adopt $(\lambda x A(x))a$ as an abbreviation for this formula. Clearly, the $\lambda$–expressions as such remain singular terms and can still be equated, that is, $\lambda x A(x) = \lambda x B(x)$ is a completely legitimate statement saying that the respective classes are the same (i.e., are co-extensional).

Let $A(x)$ now mean that $A$ possibly contains some free occurrences of $x$. Thus, $A(x)$ may contain no occurrences of $x$ at all; in such a case $A(x)$ is just $A$, and the corresponding lambda-expression $\lambda x A$ means simply "the class of all $x$ such that $A$".

We then assume the following basic principle for $\lambda$, $\in$ and sentential identity (the $\equiv$-version of *$\lambda$–conversion* or *$\beta\eta$–equality*):

$(\beta\eta)$ $(\lambda x A(x))a \equiv A(x/a)$.

In particular, if $x$ does not occur in $A(x)$, then $(\lambda x A)a \equiv A$.

Clearly, if we allow an unrestricted applicability of lambda-terms, then $(\beta\eta)$ turns out to be just an instance of the Comprehension Axiom, and, as a result, Russell's paradox is easily derivable[7]. Therefore, to exclude expressions of the form $(\lambda x A(x))\lambda x A(x)$, $(\beta\eta)$ has to be suitably restricted as, e.g., in various versions of *typed lambda calculus*. This can be done in a way analogous to a system of simple types by introducing the notion of a *well-typed $\lambda$–expression*. We assume two kinds of types which can be assigned to expressions of our language: $\sigma$ (the base type) and $\omega \mapsto \tau$ (where $\omega$ and $\tau$ are any types). Then we define recursively:

DEFINITION 11. *(Well-typed expressions)*

**Base case.** *Any individual variable, individual constant, or well-formed formula with no occurrences of the $\lambda$–operator is well-typed and has type $\sigma$.*

**Abstraction formation.** *For every expression $M$ of type $\tau$ and every variable $x$, the (well-formed) term $\lambda x M$ is well-typed and has type $\sigma \mapsto \tau$.*

**Application.** *If $M$ is well-typed of type $\omega \mapsto \tau$ and $N$ is well-typed of type $\omega$, then $(M)N$ is well-typed and has type $\tau$.*

**Extremal clause.** *No expression is well-typed unless it is obtained from either of the three clauses above.*

Here we need not delve to much into the specifics of various typed lambda calculi; it is enough just to require that in $(\beta\eta)$ all expressions must be

[7]Let $A(x)$ be $x \notin x$, and consider the term $t = \lambda x(x \notin x)$. Then, by $(\beta\eta)$, $t \in \lambda x(x \notin x) \equiv t \notin t$, and by definition of $t$, $t \in \lambda x(x \notin x) \equiv t \in t$. In pure "bracket notation": $(\lambda x(\sim(x)x))\lambda x(\sim(x)x) \equiv \sim(\lambda x(\sim(x)x))\lambda x(\sim(x)x)$.

well-typed. We mark the principle subject to this requirement as $(\beta\eta)^\tau$. Moreover, it is not necessary to define more exactly the kind of set-theory for the predicate $\in$ which is implicit in any formula of the form $(\lambda x A(x))a$. We again just assume that this predicate should be such that $(\beta\eta)^\tau$ holds for it.

Let system $\mathbf{PCI}+(\beta\eta)^\tau$ be obtained from **PCI** by extending its language with the $\in$–predicate and the $\lambda$–operator both subject to $(\beta\eta)^\tau$. Then we have the following theorem:

THEOREM 12. $(FA)$ *is provable in* $\mathbf{PCI}+(\beta\eta)^\tau$.

PROOF. We have the following outline of a proof:

1. $A \wedge B$ (*Assumption*)
2. $A$ (1, $\wedge$-*Elimination*)
3. $B$ (1, $\wedge$-*Elimination*)
4. $A \rightarrow \lambda x(\sim A) = \lambda x(x \neq x)$ ($\mathbf{PCI}+(\beta\eta)^\tau$)
5. $B \rightarrow \lambda x(\sim B) = \lambda x(x \neq x)$ ($\mathbf{PCI}+(\beta\eta)^\tau$)
6. $\lambda x(\sim A) = \lambda x(x \neq x)$ (2, 4, *MP*)
7. $\lambda x(\sim B) = \lambda x(x \neq x)$ (3, 5, *MP*)
8. $\lambda x(\sim A) = \lambda x(\sim B)$ (6, 7, *Transitivity of* $=$)
9. $\lambda x(\sim A) = \lambda x(\sim B) \rightarrow (\lambda x(\sim A))a \equiv (\lambda x(\sim B))a$ (Ax4)
10. $(\lambda x(\sim A))a \equiv (\lambda x(\sim B))a$ (8, 9, *MP*)
11. $\sim A \equiv \sim B$ (10, $(\beta\eta)^\tau$, *Transitivity of* $\equiv$)
12. $\sim A \equiv \sim B \leftrightarrow A \equiv B$ (Ax2)
13. $A \equiv B$ (11, 12, *MP*)

Thus, by the Deduction Theorem, $\vdash (A \wedge B) \rightarrow (A \equiv B)$. *Mutatis mutandis* we get also $\vdash (\sim A \wedge \sim B) \rightarrow (A \equiv B)$. Hence, $\vdash (A \leftrightarrow B) \rightarrow (A \equiv B)$. ■

Now one can remark (and even find it problematic) that $\mathbf{PCI}+(\beta\eta)^\tau$ is formulated not in a pure logical language, but in fact represents an applied theory of the $\in$–predicate (some kind of set theory). For such a critical observer we propose to consider another approach to the $\lambda$–operator, where it serves as a device which allows to manipulate arbitrary formulas into predicates. If we take $\lambda x A(x)$ as the characteristic function of the interpretation of the predicate $A(x)$, then $\lambda x A(x)$ may be treated not as a singular term which denotes a function, but as a predicate. If $x$ is not among the free variables of $A$, then the interpretation of a predicate $\lambda x A(x)$ in a model is the individual domain of the model, if $A$ is true in the model, and otherwise it is the empty set. In any case, an expression $(\lambda x A(x))a$ then is a formula. This kind of abstracting a predicate from an open formula is called *predicate abstraction* and was introduced into modal logic by Stalnaker and Thomason in the late 1960ies, see [13, Chapters 9 and 10], the references given there, and [12, Section 3].

The lambda-expression $\lambda x A(x)$ stands then for "the property of being (an $x$ such that $x$ is) $A$". This is a somewhat "intensional" reading which strictly parallels the set-theoretic interpretation taken above[8]. The formula $(\lambda x A(x))a$ can be read as "The property of being (an $x$ such that $x$ is) $A$ applies to $a$" or "$a$ possesses the property $\lambda x A(x)$". If $char_A$ is the characteristic function of the interpretation of $A(x)$, the formula may be understood as saying that $char_A(a) = 1$. Thus, the $\lambda$–operator is now treated as a predicate-forming operator and *not* as a term-forming operator.

The schema $(\beta\eta)$ can be taken as it stands without any typing machinery, for the very distinction between singular terms and predicates outlaws the unwanted expressions like $(\lambda x A(x))\lambda x A(x)$. The principle $(\beta\eta)$ may be read then as saying that "The property of being (an $x$ such that $x$ is) $A$ applies to $a$" describes the same situation as "$a$ is $A$". Predicate abstraction enables scope distinctions in the presence of non-rigid singular terms. We here assume rigidly designating individual constants.

However, considering $\lambda$ as a predicate-forming operator has as a result that the lambda-expressions so conceived cannot be equated in a first-order language. Such an equating is crucial for constructing the "$\lambda$–version" of the slingshot argument presented in Theorems 9 and 10. Nevertheless, we may consider a second-order extension of **PCI**.

Let the system $\mathbf{PCI}^2$+$(\beta\eta)$ be second-order **PCI** extended by the predicate abstracting $\lambda$-operator subject to $(\beta\eta)$, the second-order identity predicate $=^2$, and the following second-order version Ax4$^2$ of Axiom Ax4:

Ax4$^2$. $(X =^2 Y) \rightarrow (X(a) \equiv Y(a))$,

for second-order variables $X$, $Y$. Then we have the following theorem:

THEOREM 13. $(FA)$ *is provable in* $\mathbf{PCI}^2$+$(\beta\eta)$.

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[8]By the way, Carnap apparently considers both interpretations on a par when he introduces "abstraction expressions ('$(\lambda x)(..x..)$', 'the property (or class) of those $x$ which are such that $..x..$')" [6, p. 3].

Proof. The proof is *mutatis mutandis* as in Theorem 12 with the difference that on steps 4–9 the main sign of the (first-order) identity ($=$) has to be changed to the second-order one ($=^2$), and the justifications of these steps have to be modified correspondingly. ∎

## 7. Non-Fregean logic and indefinite descriptions

The construction of lambda-terms suggests the possibility to consider an enrichment of **PCI** not with lambda-expressions, but with indefinite descriptions. Indeed, we may well dissociate "the property of being (an $x$ such that $x$ is) $A$" and concentrate ourselves on "an $x$ such that ($x$ is) $A$" itself. Such an expression can be formalized by means of some sort of operator for indefinite descriptions. One can find in the literature several different accounts of indefinite descriptions (for an overview of these approaches and the corresponding operators consult, e.g., [18]). One of the best formally elaborated theories of such an operator is the famous $\varepsilon$–formalism introduced by David Hilbert, see [1], [19].

Thus, if $A(x)$ is a formula possibly containing free occurrences of $x$, then $\varepsilon x A(x)$ is a singular term for "an $x$, such that ($x$ is) $A$". If $x$ does not occur in $A(x)$, then $A(x)$ is just $A$, and $\varepsilon x A$ means "an $x$, such that $A$".

Now it is natural to suppose that an object, such that it is $A$ and is equal to $a$, is exactly $a$, if and only if $a$ is $A$. In other words, in the case when we consider some concrete object $x = a$, we can accept for indefinite descriptions the principles strictly analogous to ($\bullet$) and ($\bullet'$) just by changing $\iota$ to $\varepsilon$. Consequently, one immediately obtains the versions of the slingshot argument as formulated in Theorems 7 and 8 by using $\varepsilon$-expressions instead of $\iota$-expressions.

However, in the "indefinite analogues" of ($\bullet$) and ($\bullet'$) the operator of description turns out to be explicitly submitted to an additional condition of uniqueness which is in fact characteristic of definite descriptions. As to indefinite descriptions, the uniqueness of a described object can only be considered a particular case and cannot be taken as a general condition. Indeed, as it was observed by Russell, "the only thing that distinguishes 'the so-and-so' from 'a so-and-so' is the implication of uniqueness" [33, p. 176].

It could also be possible to construct a slingshot argument within non-Fregean Logic enriched by a kind of an (indefinite) description operator without the uniqueness condition, provided this operator possesses some peculiar features. Namely, consider the operator $\kappa$ such that if $A(x)$ is a formula possibly containing free occurrences of $x$, then $\kappa x A(x)$ is a singular term. We first require that if $A(x)$ is a false sentence, then $\kappa x A(x)$ should mean exactly the same as $\kappa x(x \neq x)$ (call this feature *F-property*). And second we assume the following scheme:

$$(\kappa^{=})\ \ \kappa x A(x) = \kappa x B(x) \rightarrow \exists x(A(x) \equiv B(x)).$$

That is to say, if $\kappa x A(x)$ and $\kappa x B(x)$ denote the same object(s), then there exists at least one object $x$ such that $A(x)$ and $B(x)$ denote the same situation.

Now let **PCI**+$(\kappa^{=})$ be **PCI** enriched with the $\kappa$–operator subject to $(\kappa^{=})$. Then we have the following theorem:

THEOREM 14. $(FA)$ *is provable in* **PCI**+$(\kappa^{=})$.

PROOF. Consider the following outline of a proof:

1. $A \wedge B$ (*Assumption*)
2. $A$ (1, $\wedge$–*Elimination*)
3. $B$ (1, $\wedge$–*Elimination*)
4. $A \rightarrow \kappa x({\sim}A) = \kappa x(x \neq x)$ (*F-property*)
5. $B \rightarrow \kappa x({\sim}B) = \kappa x(x \neq x)$ (*F-property*)
6. $\kappa x({\sim}A) = \kappa x(x \neq x)$ (2, 4, *MP*)
7. $\kappa x({\sim}B) = \kappa x(x \neq x)$ (3, 5, *MP*)
8. $\kappa x({\sim}A) = \kappa x({\sim}B)$ (6, 7, *Transitivity of* $=$)
9. $\kappa x({\sim}A) = \kappa x({\sim}B) \rightarrow \exists x({\sim}A \equiv {\sim}B)$ $(\kappa^{=})$
10. $\exists x({\sim}A \equiv {\sim}B)$ (8, 9, *MP*)
11. ${\sim}A \equiv {\sim}B$ (10, $\exists$–*Elimination*)
12. ${\sim}A \equiv {\sim}B \leftrightarrow A \equiv B$ (Ax2)
13. $A \equiv B$ (11, 12, *MP*)

Again, by the Deduction Theorem, $\vdash (A \wedge B) \rightarrow (A \equiv B)$. *Mutatis mutandis* we get also $\vdash ({\sim}A \wedge {\sim}B) \rightarrow (A \equiv B)$. Hence, $\vdash (A \leftrightarrow B) \rightarrow (A \equiv B)$. ■

Thus, extending **NFL** with an operator of the $\kappa$–type also exerts a destructive effect on the connective of sentential identity. However, a possible informal reading (as well as the precise semantical interpretation) of the kappa-operator remains not quite clear. It obviously cannot be Hilbert's epsilon-operator, since $(\kappa^{=})$ turns out to be implausible if we just change $\kappa$ to $\varepsilon$. Taken the standard understanding of $\varepsilon x$ as "some (particular) $x$", it is clear that even if "*some* (particular) man that is ill" and "*some* (particular) man that is tall" happen to denote the same person, it would hardly probable to conclude on this ground that to be ill and to be tall constitute the same situation for anyone.

Nevertheless, in the literature it has been repeatedly stressed that an indefinite description can ambiguously stand not only for "some (particular)", but also for "any (whatever)" object. As George Wilson put it: "In fact, the form of words $\ulcorner$A(n)$\phi$ is $\psi\urcorner$ seems to be generally ambiguous between ... *Something which is* a(n) $\phi$ is $\psi$ ... and ... *Anything which is* a(n) $\phi$ is $\psi$" [39, p. 50]. Wilson also argues that a pure quantificational interpretation of indefinite descriptions reveals certain difficulties as well.

By taking this into account, it seems more natural to interpret the $\kappa$–operator in the second sense, i.e., as some kind of "any"–phrase (from "$\kappa\acute{\alpha}\theta\varepsilon$" in Greek), which does not obligatorily have a pure quantificational meaning. Under such an informal understanding ($\kappa^{=}$) looks plausible enough. But many questions concerning this operator (first of all, its precise syntactical and semantical elaboration) are still open.

## 8. Concluding remarks

The slingshot argument has caused much controversy especially on the part of fact-theorists and adherents of situations, states of affairs or other fact-like entities, who try to discredit its probative force in one way or another. In particular, opponency by Barwise and Perry from the standpoint of their situation semantics has gained general attention.

In this respect the enterprise undertaken by Wójtowicz is quite creditable. Whereas the formal reconstruction of the slingshot argument in the style of Neale (see Section 2.2) focuses on the inferential properties of the $\iota$–operator, the reconstruction in the style of Wójtowicz makes use of the properties of sentential, descriptive identity as axiomatized in **PCI** and in $\mathbf{WBQ}^{\iota}$ or **WBQ** extended by the $\lambda$–operator (or even the $\kappa$-operator). In summary, it seems fair to say that despite a multifarious and sometimes sophisticated criticism, the slingshot argument presents a powerful and lucid justification of the view that sentences do signify truth values.

Nevertheless, this does not at all show that an ontology and semantics of situations (facts, states of affairs, etc.) is not worthy of investigation or even is technically infeasible, see, for example, [3], [40]. Still, one has to be very cautious when combining such theories with definite descriptions or lambda-expressions. As Wójtowicz [42, p. 190] emphasizes, "the description or abstraction operators must have their interpretation in such an ontology".

Gödel [17] already explained that the problem posed by his version of the slingshot argument can be solved by adopting Russell's contextual theory of definite descriptions. In the case of the $\lambda$–abstractor, Wójtowicz suggests to consider a Russellian theory of $\lambda$–terms in which an expression $A(\lambda x B(x))$ is regarded as an abbreviation of the sentence $\exists x \forall y((y \in x \leftrightarrow B(y)) \wedge A(x))$.[9] We can observe that this move in fact blocks the derivation from the proof of Theorems 9, 10, and 12, because we are no longer dealing with a substitution of singular terms licensed by axiom Ax4.

The version of the slingshot argument using predicate abstraction, however, is *not* blocked in this way, because now we are using $\lambda$-*predicates* and, instead of an application of Ax4, we have an application of it's second-order version Ax$4^2$. In other words, the slingshot argument based on predicate abstraction cannot be circumvented by an appeal to a "Russellian" theory of $\lambda$–*terms*. In this sense $(\beta\eta)$ and Ax$4^2$ pinpoint minimal assumptions about descriptive sentential identity needed to derive the Fregean Axiom.

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