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Nonclassical logics corner

Embedding from multilattice logic into classical logic and vice versa

NORIHIRO KAMIDE, *Department of Information and Electronic Engineering, Teikyo University, Toyosatodai 1-1, Utsunomiya, Tochigi 320-8551, Japan. E-mail: drnkamide08@kpd.biglobe.ne.jp*

YAROSLAV SHRAMKO, *Department of Philosophy, Kryvyi Rih State Pedagogical University, 50086 Kryvyi Rih, Ukraine. E-mail: shramko@rocketmail.com*

Abstract

This article presents some theorems for syntactic and semantic embeddings of a Gentzen-type sequent calculus ML_n for multilattice logic into a Gentzen-type sequent calculus LK for classical logic and vice versa. These embedding theorems are used to prove cut-elimination, decidability and completeness theorems for ML_n, as well as a modified Craig interpolation theorem. Some of these results are then extended to the first-order system FML_n with implications and co-implications.

Keywords: Multilattice logic, embbeding theorem, completeness theorem, sequent calculus, interpolation theorem, cut-elimination theorem.

1 Introduction: a logic for multilattices and its metaproperties

The bilattices introduced by Ginsberg in [6] and [7] are generally acknowledged as a useful and highly effective tool for the semantic and algebraic analysis of many-valued logics in which truth values are naturally ordered according to their 'truth-content' *and* informativity, see, e.g. [5].

Arieli and Avron [2] highlighted the proof-theoretic importance of bilattices by introducing the notion of a *logical bilattice* and constructing the corresponding proof systems. For a bilattice $\mathcal{B}$, they considered the notions of a *prime bifilter* and *ultrabifilter*, each determined by *both* bilattice orderings, and then defined the logical bilattice (ultralogical bilattice) as a pair $(\mathcal{B},F)$, where $\mathcal{B}$ is a bilattice (bilattice with conflation) and $\mathcal{F}$ is a prime bifilter (ultrabifilter) on $\mathcal{B}$. These logical bilattices were used then 'for defining logics in a way which is completely analogous to the way Boolean algebras and prime filters are used in classical logic' [2, pp. 30–31]. The logics for (ultra)logical bilattices involve operations with respect to both bilattice-orderings. The basic system for capturing the bilattice meets, joins, and inversions is described in [2, pp. 37–40] in the form of a Gentzen-type sequent calculus GBL.

In [19], the theory of logical bilattices was generalized by introducing the notion of a *logical multilattice* based on a lattice with n ordering relations. These relations can represent various possible characterizations of given truth values such as information, truth, falsity, constructivity, (un)certainty, modality and other kinds of 'adverbial qualifications'. Roughly speaking, a logical multilattice $(\mathcal{M}_n,\mathcal{F}_n)$ is a multilattice (n-lattice) $\mathcal{M}_n$ equipped with a *prime multifilter* (n-filter) $\mathcal{F}_n$. If $\mathcal{F}_n$ is defined as an *ultramultifilter*, the multilattice turns out to be *ultralogical*. (Ultra)logical multilattices allow one to deal simultaneously with n pairs of meets and joins involving the corresponding inversion operations.

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At the level of an object language, this algebraic framework can be grasped by a language $\mathcal{L}_n$ with exactly n connectives for negations, conjunctions and disjunctions determined by the corresponding orderings in a given multilattice:

$$\mathcal{L}_n : A ::= p \,|\, \sim_1 A \,|\, \ldots \,|\, \sim_n A \,|\, A \wedge_1 A \,|\, \ldots \,|\, A \wedge_n A \,|\, A \vee_1 A \,|\, \ldots \,|\, A \vee_n A.$$

It is a standard operation to introduce a valuation function for $\mathcal{L}_n$-sentences with respect to a given n-lattice, and to define an entailment relation between $\mathcal{L}_n$-sentences by taking the elements from an ultramultifilter as designated truth values.

For a proof-theoretic characterization of this relation a Gentzen-type sequent calculus GML$_n$ has been formulated in [19] for reasoning with ultralogical multilattices of an arbitrary dimension n, analogous to the GBL system for (ultra)logical bilattices from [2]. The soundness and completeness of GML$_n$ with respect to the logical n-lattices (as well as cut-elimination) can be proved by defining a canonical valuation for any logical multilattice $(\mathcal{M}_n, \mathcal{F}_n)$ in such a way that for every sequent $\Gamma \Rightarrow \Delta$ all sentences from Γ take values from $\mathcal{F}_n$, and all sentences from Δ take values beyond $\mathcal{F}_n$ (cf. the proof of Theorem 3.7 in [2]).

In this article, we examine a very different way of establishing these metatheoretical properties of the multilattice logic. Namely, we define suitable translation operations between multilattice logic and classical logic, and thus derive an embedding technique to show that one logic possesses some important properties that are known to be inherent in another logic.

The remainder of this article can be summarized as follows.

In Section 2, some theorems for the syntactic embedding of multilattice logic into classical logic and vice versa are presented. First, the ML$_n$ and LK systems are introduced (ML$_n$ being a slight variation of GML$_n$ from [19]), and a translation function f from ML$_n$ into LK is defined. Next, a weak theorem for the syntactic embedding of ML$_n$ into LK based on f is stated. The cut-elimination and decidability theorems for ML$_n$ are obtained using this weak embedding theorem. A strong theorem for the syntactic embedding of ML$_n$ into LK is also obtained. Next, a translation function g from LK into ML$_n$ is defined, and similar weak and strong theorems for the embedding of LK into ML$_n$ are derived in a similar way based on g.

In Section 3, some theorems for the semantic embedding of ML$_n$ into LK and vice versa are presented. A classical-like valuation semantics for ML$_n$ and standard semantics for LK are introduced. Two theorems for the semantic embedding of ML$_n$ into LK and vice versa are then shown. The completeness theorem with respect to the valuation semantics of ML$_n$ is obtained using both the syntactical and semantical embedding theorems of ML$_n$ into LK.

In Section 4, a modified Craig interpolation theorem is proved for ML$_n$ using the syntactical embedding theorem of ML$_n$ into LK. As a corollary, the Maksimova separation theorem (Maksimova's principle of variable separation) for ML$_n$ is also obtained.

In Section 5, some of these results are extended to the first-order system FML$_n$ with implications and co-implications.

Some remarks summarizing the embedding-based proof methods, as well as certain concluding observations, are given in Sections 6 and 7.

2 Syntactical embedding and cut-elimination

Consider the language $\mathcal{L}_n$ generally defined as above, where n (>1) is a positive integer determined by some n-lattice. That is, the *formulas* of the n-lattice logic are constructed from countably many propositional variables by the logical connectives $\wedge_j$, $\vee_j$ and $\sim_j$, for every $j \leq n$. In the following, we use small letters $p, q, \ldots$ to denote propositional variables, Greek small letters $\alpha, \beta, \ldots$ to denote formulas and Greek capital letters $\Gamma, \Delta, \ldots$ to represent finite (possibly empty) sets of formulas.

A *sequent* is an expression of the form $\Gamma \Rightarrow \Delta$. An expression $L \vdash \Gamma \Rightarrow \Delta$ means that the sequent $\Gamma \Rightarrow \Delta$ is provable in a sequent calculus L. A Gentzen-type sequent calculus ML_n for n-lattice logic,which was originally introduced in [19], can be defined as follows:

Definition 2.1 (ML_n)
Let n (>1) be the positive integer determined by n-lattice, and j, k be any positive integers with $j, k \leq n$ and $j \neq k$.

The initial sequents of ML_n are of the following form, for any propositional variable p,

$$p \Rightarrow p \qquad\qquad {\sim_j} p \Rightarrow {\sim_j} p.$$

The structural inference rules of ML_n are of the form:

$$\frac{\Gamma \Rightarrow \Delta, \alpha \quad \alpha, \Sigma \Rightarrow \Pi}{\Gamma, \Sigma \Rightarrow \Delta, \Pi} \text{ (cut)}$$

$$\frac{\Gamma \Rightarrow \Delta}{\alpha, \Gamma \Rightarrow \Delta} \text{ (we-left)} \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \alpha} \text{ (we-right).}$$

The logical inference rules of ML_n are of the form:

$$\frac{\alpha, \beta, \Gamma \Rightarrow \Delta}{\alpha \wedge_j \beta, \Gamma \Rightarrow \Delta} \text{ }(\wedge_j\text{left}) \qquad \frac{\Gamma \Rightarrow \Delta, \alpha \quad \Gamma \Rightarrow \Delta, \beta}{\Gamma \Rightarrow \Delta, \alpha \wedge_j \beta} \text{ }(\wedge_j\text{right})$$

$$\frac{\alpha, \Gamma \Rightarrow \Delta \quad \beta, \Gamma \Rightarrow \Delta}{\alpha \vee_j \beta, \Gamma \Rightarrow \Delta} \text{ }(\vee_j\text{left}) \qquad \frac{\Gamma \Rightarrow \Delta, \alpha, \beta}{\Gamma \Rightarrow \Delta, \alpha \vee_j \beta} \text{ }(\vee_j\text{right})$$

$$\frac{{\sim_j}\alpha, \Gamma \Rightarrow \Delta \quad {\sim_j}\beta, \Gamma \Rightarrow \Delta}{{\sim_j}(\alpha \wedge_j \beta), \Gamma \Rightarrow \Delta} \text{ }({\sim_j}\wedge_j\text{left}) \qquad \frac{\Gamma \Rightarrow \Delta, {\sim_j}\alpha, {\sim_j}\beta}{\Gamma \Rightarrow \Delta, {\sim_j}(\alpha \wedge_j \beta)} \text{ }({\sim_j}\wedge_j\text{right})$$

$$\frac{{\sim_j}\alpha, {\sim_j}\beta, \Gamma \Rightarrow \Delta}{{\sim_j}(\alpha \vee_j \beta), \Gamma \Rightarrow \Delta} \text{ }({\sim_j}\vee_j\text{left}) \qquad \frac{\Gamma \Rightarrow \Delta, {\sim_j}\alpha \quad \Gamma \Rightarrow \Delta, {\sim_j}\beta}{\Gamma \Rightarrow \Delta, {\sim_j}(\alpha \vee_j \beta)} \text{ }({\sim_j}\vee_j\text{right})$$

$$\frac{\alpha, \Gamma \Rightarrow \Delta}{{\sim_j}{\sim_j}\alpha, \Gamma \Rightarrow \Delta} \text{ }({\sim_j}{\sim_j}\text{left}) \qquad \frac{\Gamma \Rightarrow \Delta, \alpha}{\Gamma \Rightarrow \Delta, {\sim_j}{\sim_j}\alpha} \text{ }({\sim_j}{\sim_j}\text{right})$$

$$\frac{{\sim_k}\alpha, {\sim_k}\beta, \Gamma \Rightarrow \Delta}{{\sim_k}(\alpha \wedge_j \beta), \Gamma \Rightarrow \Delta} \text{ }({\sim_k}\wedge_j\text{left}) \qquad \frac{\Gamma \Rightarrow \Delta, {\sim_k}\alpha \quad \Gamma \Rightarrow \Delta, {\sim_k}\beta}{\Gamma \Rightarrow \Delta, {\sim_k}(\alpha \wedge_j \beta)} \text{ }({\sim_k}\wedge_j\text{right})$$

$$\frac{{\sim_k}\alpha, \Gamma \Rightarrow \Delta \quad {\sim_k}\beta, \Gamma \Rightarrow \Delta}{{\sim_k}(\alpha \vee_j \beta), \Gamma \Rightarrow \Delta} \text{ }({\sim_k}\vee_j\text{left}) \qquad \frac{\Gamma \Rightarrow \Delta, {\sim_k}\alpha, {\sim_k}\beta}{\Gamma \Rightarrow \Delta, {\sim_k}(\alpha \vee_j \beta)} \text{ }({\sim_k}\vee_j\text{right})$$

$$\frac{\Gamma \Rightarrow \Delta, \alpha}{{\sim_k}{\sim_j}\alpha, \Gamma \Rightarrow \Delta} \text{ }({\sim_k}{\sim_j}\text{left}) \qquad \frac{\alpha, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, {\sim_k}{\sim_j}\alpha} \text{ }({\sim_k}{\sim_j}\text{right}).$$

We first make the following remarks.

(1) The system ML_n was originally called GML_n in [19].
(2) The present system ML_n has some minor modifications with respect to the original, e.g. the forms of initial sequents and structural inference rules. However, the provability of the system is unchanged.

(3) Sequents of the form $\alpha \Rightarrow \alpha$ for any formula α are provable in ML_n. This fact can be shown by induction on α.
(4) ML_n can simulate the classical negation connective $\neg$ by $\sim_k\sim_j$, e.g. the excluded middle-like sequent of the form $\Rightarrow \sim_k\sim_j\alpha\vee_j\alpha$ is provable in ML_n.

A Gentzen-type sequent calculus LK for the $\rightarrow$-free fragment of propositional classical logic is introduced below. Formulas of LK are constructed from countably many propositional variables and logical connectives $\wedge$, $\vee$ and $\neg$.

DEFINITION 2.2 (LK)
The initial sequents of LK are of the following form, for any propositional variable p,

$$p \Rightarrow p.$$

The structural inference rules of LK are the same as that of ML_n.
The logical inference rules of LK are of the form:

$$\frac{\alpha,\beta,\Gamma\Rightarrow\Delta}{\alpha\wedge\beta,\Gamma\Rightarrow\Delta}\ (\wedge\text{left}) \qquad \frac{\Gamma\Rightarrow\Delta,\alpha \quad \Gamma\Rightarrow\Delta,\beta}{\Gamma\Rightarrow\Delta,\alpha\wedge\beta}\ (\wedge\text{right})$$

$$\frac{\alpha,\Gamma\Rightarrow\Delta \quad \beta,\Gamma\Rightarrow\Delta}{\alpha\vee\beta,\Gamma\Rightarrow\Delta}\ (\vee\text{left}) \qquad \frac{\Gamma\Rightarrow\Delta,\alpha,\beta}{\Gamma\Rightarrow\Delta,\alpha\vee\beta}\ (\vee\text{right})$$

$$\frac{\Gamma\Rightarrow\Delta,\alpha}{\neg\alpha,\Gamma\Rightarrow\Delta}\ (\neg\text{left}) \qquad \frac{\alpha,\Gamma\Rightarrow\Delta}{\Gamma\Rightarrow\Delta,\neg\alpha}\ (\neg\text{right}).$$

In what follows, we introduce an LK-translation function for formulas of ML_n, and by using this translation, we show some theorems for embedding of ML_n into LK. A similar translation has been used by Gurevich [8], Rautenberg [18] and Vorob'ev [22] to embed Nelson's constructive logic [1, 17] into intuitionistic logic. Some similar translations have also been used by Kamide [9, 12–14] to embed some paraconsistent logics into classical logic.

DEFINITION 2.3
Let n (>1) be the positive integer determined by n-lattice, and let j,k be any positive integers with $j,k\leq n$ and $j\neq k$. We fix a set Φ of propositional variables and define for every j the sets $\Phi^j := \{p^j \mid p\in\Phi\}$ of propositional variables. The language $\mathcal{L}_{\mathrm{ML}}$ of ML_n is defined using Φ, $\wedge_j, \vee_j$ and $\sim_j$ (cf. general definition of language $\mathcal{L}_n$ above). The language $\mathcal{L}_{\mathrm{LK}}$ of LK is defined using Φ, $\Phi^1,\ldots,\Phi^n$, $\wedge$, $\vee$ and $\neg$. A mapping f from $\mathcal{L}_{\mathrm{ML}}$ to $\mathcal{L}_{\mathrm{LK}}$ is defined inductively by:

(1) For any $p\in\Phi$, for any $j\leq n$, $f(p):=p$ and $f(\sim_j p):=p^j\in\Phi^j$,
(2) $f(\alpha\wedge_j\beta):=f(\alpha)\wedge f(\beta)$,
(3) $f(\alpha\vee_j\beta):=f(\alpha)\vee f(\beta)$,
(4) $f(\sim_j\sim_j\alpha):=f(\alpha)$,
(5) $f(\sim_j(\alpha\wedge_j\beta)):=f(\sim_j\alpha)\vee f(\sim_j\beta)$,
(6) $f(\sim_j(\alpha\vee_j\beta)):=f(\sim_j\alpha)\wedge f(\sim_j\beta)$,
(7) $f(\sim_k\sim_j\alpha):=\neg f(\alpha)$,
(8) $f(\sim_k(\alpha\wedge_j\beta)):=f(\sim_k\alpha)\wedge f(\sim_k\beta)$,
(9) $f(\sim_k(\alpha\vee_j\beta)):=f(\sim_k\alpha)\vee f(\sim_k\beta)$.

An expression $f(\Gamma)$ denotes the result of replacing every occurrence of a formula α in Γ by an occurrence of $f(\alpha)$. Analogous notation is used for the other mapping g discussed later.

We now show a weak theorem for syntactic embedding of ML_n into LK. A similar theorem was shown for a trilattice logic as Theorem 3.4 in [14].

THEOREM 2.4 (Weak syntactical embedding from ML_n into LK)
Let Γ, Δ be sets of formulas in $\mathcal{L}_{\mathrm{ML}}$, and f be the mapping defined in Definition 2.3.

(1) If $\mathrm{ML}_n \vdash \Gamma \Rightarrow \Delta$, then $\mathrm{LK} \vdash f(\Gamma) \Rightarrow f(\Delta)$.
(2) If $\mathrm{LK} - (\mathrm{cut}) \vdash f(\Gamma) \Rightarrow f(\Delta)$, then $\mathrm{ML}_n - (\mathrm{cut}) \vdash \Gamma \Rightarrow \Delta$.

PROOF. • (1) By induction on the proofs P of $\Gamma \Rightarrow \Delta$ in ML_n. We distinguish the cases according to the last inference of P, and show some cases.

(1) Case ($\sim_j p \Rightarrow \sim_j p$): The last inference of P is of the form: $\sim_j p \Rightarrow \sim_j p$ for any $p \in \Phi$. In this case, we obtain $\mathrm{LK} \vdash f(\sim_j p) \Rightarrow f(\sim_j p)$, i.e. $\mathrm{LK} \vdash p^j \Rightarrow p^j$ ($p^j \in \Phi^j$), by the definition of f.
(2) Case ($\sim_j \wedge_j$left): The last inference of P is of the form:

$$\frac{\sim_j \alpha, \Gamma \Rightarrow \Delta \quad \sim_j \beta, \Gamma \Rightarrow \Delta}{\sim_j (\alpha \wedge_j \beta), \Gamma \Rightarrow \Delta}\ (\sim_j \wedge_j\text{left}).$$

By induction hypothesis, we have $\mathrm{LK} \vdash f(\sim_j \alpha), f(\Gamma) \Rightarrow f(\Delta)$ and $\mathrm{LK} \vdash f(\sim_j \beta), f(\Gamma) \Rightarrow f(\Delta)$. Then, we obtain the required fact:

$$\frac{\begin{array}{c}\vdots\\ f(\sim_j \alpha), f(\Gamma) \Rightarrow f(\Delta)\end{array} \quad \begin{array}{c}\vdots\\ f(\sim_j \beta), f(\Gamma) \Rightarrow f(\Delta)\end{array}}{f(\sim_j \alpha) \vee f(\sim_j \beta), f(\Gamma) \Rightarrow f(\Delta)}\ (\vee\text{left})$$

where $f(\sim_j \alpha) \vee f(\sim_j \beta)$ coincides with $f(\sim_j (\alpha \wedge_j \beta))$ by the definition of f.
(3) Case ($\sim_k \sim_j$left): The last inference of P is of the form:

$$\frac{\Gamma \Rightarrow \Delta, \alpha}{\sim_k \sim_j \alpha, \Gamma \Rightarrow \Delta}\ (\sim_k \sim_j\text{left}).$$

By induction hypothesis, we have $\mathrm{LK} \vdash f(\Gamma) \Rightarrow f(\Delta), f(\alpha)$. Then, we obtain the required fact:

$$\frac{\begin{array}{c}\vdots\\ f(\Gamma) \Rightarrow f(\Delta), f(\alpha)\end{array}}{\neg f(\alpha), f(\Gamma) \Rightarrow f(\Delta)}\ (\neg\text{left})$$

where $\neg f(\alpha)$ coincides with $f(\sim_k \sim_j \alpha)$ by the definition of f.

• (2) By induction on the proofs Q of $f(\Gamma) \Rightarrow f(\Delta)$ in $\mathrm{LK} - (\mathrm{cut})$. We distinguish the cases according to the last inference of Q, and show only the following cases.

(1) Case ($\neg$left): The last inference of Q is of the form:

$$\frac{f(\Gamma) \Rightarrow f(\Delta), f(\alpha)}{f(\sim_k \sim_j \alpha), f(\Gamma) \Rightarrow f(\Delta)}\ (\neg\text{left})$$

where $f(\sim_k\sim_j\alpha)$ coincides with $\neg f(\alpha)$ by the definition of f. By induction hypothesis, we have $\mathrm{ML}_n-(\mathrm{cut})\vdash\Gamma\Rightarrow\Delta,\alpha$. We thus obtain the required fact:

$$\dfrac{\begin{array}{c}\vdots\\ \Gamma\Rightarrow\Delta,\alpha\end{array}}{\sim_k\sim_j\alpha,\Gamma\Rightarrow\Delta}\ (\sim_k\sim_j\text{left}).$$

(2) Case ($\wedge$right): The last inference of Q is ($\wedge$right).

(a) Subcase (1): The last inference of Q is of the form:

$$\dfrac{f(\Gamma)\Rightarrow f(\Delta),f(\alpha)\quad f(\Gamma)\Rightarrow f(\Delta),f(\beta)}{f(\Gamma)\Rightarrow f(\Delta),f(\alpha\wedge_j\beta)}\ (\wedge\text{right})$$

where $f(\alpha\wedge_j\beta)$ coincides with $f(\alpha)\wedge f(\beta)$ by the definition of f. By induction hypothesis, we have $\mathrm{ML}_n-(\mathrm{cut})\vdash\Gamma\Rightarrow\Delta,\alpha$ and $\mathrm{ML}_n-(\mathrm{cut})\vdash\Gamma\Rightarrow\Delta,\beta$. We thus obtain the required fact:

$$\dfrac{\begin{array}{c}\vdots\\ \Gamma\Rightarrow\Delta,\alpha\end{array}\quad\begin{array}{c}\vdots\\ \Gamma\Rightarrow\Delta,\beta\end{array}}{\Gamma\Rightarrow\Delta,\alpha\wedge_j\beta}\ (\wedge_j\text{right}).$$

(b) Subcase (2): The last inference of Q is of the form:

$$\dfrac{f(\Gamma)\Rightarrow f(\Delta),f(\sim_j\alpha)\quad f(\Gamma)\Rightarrow f(\Delta),f(\sim_j\beta)}{f(\Gamma)\Rightarrow f(\Delta),f(\sim_j(\alpha\vee_j\beta))}\ (\wedge\text{right})$$

where $f(\sim_j(\alpha\vee_j\beta))$ coincides with $f(\sim_j\alpha)\wedge f(\sim_j\beta)$ by the definition of f. By induction hypothesis, we have $\mathrm{ML}_n-(\mathrm{cut})\vdash\Gamma\Rightarrow\Delta,\sim_j\alpha$ and $\mathrm{ML}_n-(\mathrm{cut})\vdash\Gamma\Rightarrow\Delta,\sim_j\beta$. We thus obtain the required fact:

$$\dfrac{\begin{array}{c}\vdots\\ \Gamma\Rightarrow\Delta,\sim_j\alpha\end{array}\quad\begin{array}{c}\vdots\\ \Gamma\Rightarrow\Delta,\sim_j\beta\end{array}}{\Gamma\Rightarrow\Delta,\sim_j(\alpha\vee_j\beta)}\ (\sim_j\vee_j\text{right}).$$

(c) Subcase (3): The last inference of Q is of the form:

$$\dfrac{f(\Gamma)\Rightarrow f(\Delta),f(\sim_k\alpha)\quad f(\Gamma)\Rightarrow f(\Delta),f(\sim_k\beta)}{f(\Gamma)\Rightarrow f(\Delta),f(\sim_k(\alpha\wedge_j\beta))}\ (\wedge\text{right})$$

where $f(\sim_k(\alpha\wedge_j\beta))$ coincides with $f(\sim_k\alpha)\wedge f(\sim_k\beta)$ by the definition of f. By induction hypothesis, we have $\mathrm{ML}_n-(\mathrm{cut})\vdash\Gamma\Rightarrow\Delta,\sim_k\alpha$ and $\mathrm{ML}_n-(\mathrm{cut})\vdash\Gamma\Rightarrow\Delta,\sim_k\beta$. We thus obtain the required fact:

$$\dfrac{\begin{array}{c}\vdots\\ \Gamma\Rightarrow\Delta,\sim_k\alpha\end{array}\quad\begin{array}{c}\vdots\\ \Gamma\Rightarrow\Delta,\sim_k\beta\end{array}}{\Gamma\Rightarrow\Delta,\sim_k(\alpha\wedge_j\beta)}\ (\sim_k\wedge_j\text{right}).$$

$\blacksquare$

Using Theorem 2.4 and the well-known cut-elimination theorem for LK, we obtain the following cut-elimination theorem for ML_n.

THEOREM 2.5 (Cut-elimination for ML_n)
The rule (cut) is admissible in cut-free ML_n, i.e. any sequent provable in ML_n can be proved without using (cut).

PROOF. Suppose $\mathrm{ML}_n \vdash \Gamma \Rightarrow \Delta$. Then, we have $\mathrm{LK} \vdash f(\Gamma) \Rightarrow f(\Delta)$ by Theorem 2.4 (1), and hence $\mathrm{LK} - (\mathrm{cut}) \vdash f(\Gamma) \Rightarrow f(\Delta)$ by the cut-elimination theorem for LK. By Theorem 2.4 (2), we obtain $\mathrm{ML}_n - (\mathrm{cut}) \vdash \Gamma \Rightarrow \Delta$. ■

Using Theorem 2.4 and the cut-elimination theorem for LK, we obtain a strong theorem for syntactic embedding of ML_n into LK. A similar theorem was shown for a trilattice logic as Theorem 3.6 in [14].

THEOREM 2.6 (Syntactical embedding from ML_n into LK)
Let Γ, Δ be sets of formulas in $\mathcal{L}_{\mathrm{ML}}$, and f be the mapping defined in Definition 2.3.

1. $\mathrm{ML}_n \vdash \Gamma \Rightarrow \Delta$ iff $\mathrm{LK} \vdash f(\Gamma) \Rightarrow f(\Delta)$.
2. $\mathrm{ML}_n - (\mathrm{cut}) \vdash \Gamma \Rightarrow \Delta$ iff $\mathrm{LK} - (\mathrm{cut}) \vdash f(\Gamma) \Rightarrow f(\Delta)$.

PROOF. • (1) ($\Longrightarrow$): By Theorem 2.4 (1). ($\Longleftarrow$): Suppose $\mathrm{LK} \vdash f(\Gamma) \Rightarrow f(\Delta)$. Then we have $\mathrm{LK} - (\mathrm{cut}) \vdash f(\Gamma) \Rightarrow f(\Delta)$ by the cut-elimination theorem for LK. We thus obtain $\mathrm{ML}_n - (\mathrm{cut}) \vdash \Gamma \Rightarrow \Delta$ by Theorem 2.4 (2). Therefore, we have $\mathrm{ML}_n \vdash \Gamma \Rightarrow \Delta$.

• (2) ($\Longrightarrow$): Suppose $\mathrm{ML}_n - (\mathrm{cut}) \vdash \Gamma \Rightarrow \Delta$. Then we have $\mathrm{ML}_n \vdash \Gamma \Rightarrow \Delta$. We then obtain $\mathrm{LK} \vdash f(\Gamma) \Rightarrow f(\Delta)$ by Theorem 2.4 (1). Therefore, we obtain $\mathrm{LK} - (\mathrm{cut}) \vdash f(\Gamma) \Rightarrow f(\Delta)$ by the cut-elimination theorem for LK. ($\Longleftarrow$): By Theorem 2.4 (2). ■

THEOREM 2.7 (Decidability for ML_n)
ML_n is decidable.

PROOF. By decidability of LK, for each α, it is possible to decide if $f(\alpha)$ is provable in LK. Then, by Theorem 2.6, ML_n is also decidable. ■

Next, we introduce a translation of LK into ML_n, and by using this translation, we show some theorems for syntactic embedding of LK into ML_n.

DEFINITION 2.8
Let $\mathcal{L}_{\mathrm{ML}}$ and $\mathcal{L}_{\mathrm{LK}}$ be the languages defined in Definition 2.3.
A mapping g from $\mathcal{L}_{\mathrm{LK}}$ to $\mathcal{L}_{\mathrm{ML}}$ is defined inductively by:

(1) For any $j \leq n$, any $p \in \Phi$, and any $p^j \in \Phi^j$, $g(p) := p$ and $g(p^j) := \sim_j p$;
(2) $g(\alpha \wedge \beta) := g(\alpha) \wedge_j g(\beta)$, where j is a fixed positive integer, such that $j \leq n$;
(3) $g(\alpha \vee \beta) := g(\alpha) \vee_j g(\beta)$, where j is a fixed positive integer, such that $j \leq n$;
(4) $g(\neg\alpha) := \sim_k \sim_j g(\alpha)$, where j and k are two fixed positive integers, such that $j, k \leq n$ and $j \neq k$.

THEOREM 2.9 (Weak syntactical embedding from LK into ML_n)
Let Γ, Δ be sets of formulas in $\mathcal{L}_{\mathrm{LK}}$, and g be the mapping defined in Definition 2.8.

(1) If $\mathrm{LK} \vdash \Gamma \Rightarrow \Delta$, then $\mathrm{ML}_n \vdash g(\Gamma) \Rightarrow g(\Delta)$.
(2) If $\mathrm{ML}_n - (\mathrm{cut}) \vdash g(\Gamma) \Rightarrow g(\Delta)$, then $\mathrm{LK} - (\mathrm{cut}) \vdash \Gamma \Rightarrow \Delta$.

PROOF. • (1) By induction on the proofs P of $\Gamma\Rightarrow\Delta$ in LK. We distinguish the cases according to the last inference of P, and show only the following cases.

(1) Case ($p^j\Rightarrow p^j$): The last inference of P is of the form: $p^j\Rightarrow p^j$ for any $p^j\in\Phi^j$. In this case, we obtain $\mathrm{ML}_n\vdash g(p^j)\Rightarrow g(p^j)$, i.e. $\mathrm{ML}_n\vdash {\sim_j} p\Rightarrow {\sim_j} p$, by the definition of g.

(2) Case ($\neg$left): The last inference of P is of the form:

$$\frac{\Gamma\Rightarrow\Delta,\alpha}{\neg\alpha,\Gamma\Rightarrow\Delta}\ (\neg\text{left}).$$

By induction hypothesis, we have $\mathrm{ML}_n\vdash g(\Gamma)\Rightarrow g(\Delta),g(\alpha)$. We then obtain the required fact:

$$\frac{\begin{matrix}\vdots\\ g(\Gamma)\Rightarrow g(\Delta),g(\alpha)\end{matrix}}{{\sim_k}{\sim_j}g(\alpha),g(\Gamma)\Rightarrow g(\Delta)}\ ({\sim_k}{\sim_j}\text{left})$$

where ${\sim_k}{\sim_j}g(\alpha)$ coincides with $g(\neg\alpha)$ by the definition of g.

• (2) By induction on the proofs Q of $g(\Gamma)\Rightarrow g(\Delta)$ in ML_n − (cut). We distinguish the cases according to the last inference of Q, and show only the following cases.

(1) Case (${\sim_k}{\sim_j}$left): The last inference of Q is of the form:

$$\frac{g(\Gamma)\Rightarrow g(\Delta),g(\alpha)}{{\sim_k}{\sim_j}g(\alpha),g(\Gamma)\Rightarrow g(\Delta)}\ ({\sim_k}{\sim_j}\text{left})$$

where ${\sim_k}{\sim_j}g(\alpha)$ coincides with $g(\neg\alpha)$ by the definition of g. By induction hypothesis, we have LK − (cut) $\vdash\Gamma\Rightarrow\Delta,\alpha$. We thus obtain the required fact:

$$\frac{\begin{matrix}\vdots\\ \Gamma\Rightarrow\Delta,\alpha\end{matrix}}{\neg\alpha,\Gamma\Rightarrow\Delta}\ (\neg\text{left}).$$

(2) Case ($\wedge_j$right): The last inference of Q is of the form:

$$\frac{g(\Gamma)\Rightarrow g(\Delta),g(\alpha)\quad g(\Gamma)\Rightarrow g(\Delta),g(\beta)}{g(\Gamma)\Rightarrow g(\Delta),g(\alpha)\wedge_j g(\beta)}\ (\wedge_j\text{right})$$

where $g(\alpha)\wedge_j g(\beta)$ coincides with $g(\alpha\wedge\beta)$ by the definition of g. By induction hypothesis, we have LK − (cut) $\vdash\Gamma\Rightarrow\Delta,\alpha$ and LK − (cut) $\vdash\Gamma\Rightarrow\Delta,\beta$. We thus obtain the required fact:

$$\frac{\begin{matrix}\vdots\\ \Gamma\Rightarrow\Delta,\alpha\end{matrix}\quad\begin{matrix}\vdots\\ \Gamma\Rightarrow\Delta,\beta\end{matrix}}{\Gamma\Rightarrow\Delta,\alpha\wedge\beta}\ (\wedge\text{right}).$$

■

THEOREM 2.10 (Syntactical embedding from LK into ML_n)
Let Γ, Δ be sets of formulas in $\mathcal{L}_{\mathrm{LK}}$, and g be the mapping defined in Definition 2.8.

(1) $\mathrm{LK}\vdash\Gamma\Rightarrow\Delta$ iff $\mathrm{ML}_n\vdash g(\Gamma)\Rightarrow g(\Delta)$.
(2) LK − (cut) $\vdash\Gamma\Rightarrow\Delta$ iff ML_n − (cut) $\vdash g(\Gamma)\Rightarrow g(\Delta)$.

PROOF. By using Theorems 2.9 and 2.5. Similar to Theorem 2.6. ■

3 Semantical embedding and completeness

In this section, we formulate a valuation semantics for ML_n (without regard to the lattice-theoretic terminology) by defining the valuation function on the two-element set of classical truth values. Namely, a *paraconsistent valuation* can be defined as follows:

DEFINITION 3.1 (Semantics for ML_n)
Let $n\,(>1)$ be the positive integer determined by n-lattice, and j,k be any positive integers with $j,k\leq n$ and $j\neq k$. A *paraconsistent valuation* v^p is a mapping from the set of all propositional variables and negated propositional variables to the set $\{t,f\}$ of truth values. The paraconsistent valuation v^p is extended to the mapping from the set of all formulas to $\{t,f\}$ by the following clauses.

(1) $v^p(\alpha\wedge_j\beta)=t$ iff $v^p(\alpha)=t$ and $v^p(\beta)=t$,
(2) $v^p(\alpha\vee_j\beta)=t$ iff $v^p(\alpha)=t$ or $v^p(\beta)=t$,
(3) $v^p(\sim_j\sim_j\alpha)=t$ iff $v^p(\alpha)=t$,
(4) $v^p(\sim_j(\alpha\wedge_j\beta))=t$ iff $v^p(\sim_j\alpha)=t$ or $v^p(\sim_j\beta)=t$,
(5) $v^p(\sim_j(\alpha\vee_j\beta))=t$ iff $v^p(\sim_j\alpha)=t$ and $v^p(\sim_j\beta)=t$,
(6) $v^p(\sim_k\sim_j\alpha)=t$ iff $v^p(\alpha)=f$,
(7) $v^p(\sim_k(\alpha\wedge_j\beta))=t$ iff $v^p(\sim_k\alpha)=t$ and $v^p(\sim_k\beta)=t$,
(8) $v^p(\sim_k(\alpha\vee_j\beta))=t$ iff $v^p(\sim_k\alpha)=t$ or $v^p(\sim_k\beta)=t$.

A sequent $\Gamma\Rightarrow\Delta$ is called *ML-valid* (denoted by $\mathrm{ML}_n\models\Gamma\Rightarrow\Delta$) iff for all paraconsistent valuation v^p, if $v^p(\alpha)=t$ for all $\alpha\in\Gamma$, then $v^p(\beta)=t$ for some $\beta\in\Delta$.

In order to show a theorem for semantic embedding of ML_n into LK, we present the standard two-valued semantics for LK.

DEFINITION 3.2 (Semantics for LK)
A valuation v is a mapping from the set of all propositional variables to the set $\{t,f\}$ of truth values. The valuation v is extended to the mapping from the set of all formulas to $\{t,f\}$ by the following clauses.

(1) $v(\alpha\wedge\beta)=t$ iff $v(\alpha)=t$ and $v(\beta)=t$,
(2) $v(\alpha\vee\beta)=t$ iff $v(\alpha)=t$ or $v(\beta)=t$,
(3) $v(\neg\alpha)=t$ iff $v(\alpha)=f$.

A sequent $\Gamma\Rightarrow\Delta$ is called *LK-valid* (denoted by $\mathrm{LK}\models\Gamma\Rightarrow\Delta$) iff for all valuation v, if $v(\alpha)=t$ for all $\alpha\in\Gamma$, then $v(\beta)=t$ for some $\beta\in\Delta$.

The following completeness theorem holds for LK. For any sets Γ and Δ of formulas,

$\mathrm{LK}\models\Gamma\Rightarrow\Delta$ iff $\mathrm{LK}\vdash\Gamma\Rightarrow\Delta$.

The following lemmas are similar to Lemmas 4.3 and 4.4 in [14] for a trilattice logic.

LEMMA 3.3
Let f be the mapping defined in Definition 2.3. For any paraconsistent valuation v^p, we can construct a valuation v such that for any formula α,

$v^p(\alpha)=t$ iff $v(f(\alpha))=t$.

PROOF. Let Φ be a set of propositional variables and for every $j\leq n$ let Φ^j be the set $\{p^j \mid p\in\Phi\}$ of propositional variables. Suppose that v^p is a paraconsistent valuation. Suppose that v is a mapping from $\Phi \cup \bigcup_{1\leq j\leq n} \Phi^j$ to $\{t,f\}$ such that

(1) $v^p(p)=t$ iff $v(p)=t$,
(2) $v^p(\sim_j p)=t$ iff $v(p^j)=t$.

Then, the lemma is proved by induction on α.
• Base step:

(1) Case when α is p, where p is a propositional variable: $v^p(p)=t$ iff $v(p)=t$ (by the assumption) iff $v(f(p))=t$ (by the definition of f).
(2) Case when α is $\sim_j p$, where p is a propositional variable: $v^p(\sim_j p)=t$ iff $v(p^j)=t$ (by the assumption) iff $v(f(\sim_j p))=t$ (by the definition of f).

• Induction step: We show some cases.

(1) Case when α is $\beta\wedge_j\gamma$: $v^p(\beta\wedge_j\gamma)=t$ iff $v^p(\beta)=t$ and $v^p(\gamma)=t$ iff $v(f(\beta))=t$ and $v(f(\gamma))=t$ (by induction hypothesis) iff $v(f(\beta)\wedge f(\gamma))=t$ iff $v(f(\beta\wedge_j\gamma))=t$ (by the definition of f).
(2) Case when α is $\sim_j\sim_j\beta$: $v^p(\sim_j\sim_j\beta)=t$ iff $v^p(\beta)=t$ iff $v(f(\beta))=t$ (by induction hypothesis) iff $v(f(\sim_j\sim_j\beta))=t$ (by the definition of f).
(3) Case when α is $\sim_j(\beta\wedge_j\gamma)$: $v^p(\sim_j(\beta\wedge_j\gamma))=t$ iff $v^p(\sim_j\beta)=t$ or $v^p(\sim_j\gamma)=t$ iff $v(f(\sim_j\beta))=t$ or $v(f(\sim_j\gamma))=t$ (by induction hypothesis) iff $v(f(\sim_j\beta)\vee f(\sim_j\gamma))=t$ iff $v(f(\sim_j(\beta\wedge_j\gamma)))=t$ (by the definition of f).
(4) Case when α is $\sim_k\sim_j\beta$: $v^p(\sim_k\sim_j\beta)=t$ iff $v^p(\beta)=f$ iff $v(f(\beta))=f$ (by induction hypothesis) iff $v(\neg f(\beta))=t$ iff $v(f(\sim_k\sim_j\beta))=t$ (by the definition of f).
(5) Case when α is $\sim_k(\beta\wedge_j\gamma)$: $v^p(\sim_k(\beta\wedge_j\gamma))=t$ iff $v^p(\sim_k\beta)=t$ and $v^p(\sim_k\gamma)=t$ iff $v(f(\sim_k\beta))=t$ and $v(f(\sim_k\gamma))=t$ (by induction hypothesis) iff $v(f(\sim_k\beta)\wedge f(\sim_k\gamma))=t$ iff $v(f(\sim_k(\beta\wedge_j\gamma)))=t$ (by the definition of f).

■

LEMMA 3.4
Let f be the mapping defined in Definition 2.3. For any valuation v, we can construct a paraconsistent valuation v^p such that for any formula α,

$$v(f(\alpha))=t \text{ iff } v^p(\alpha)=t.$$

PROOF. Similar to the proof of Lemma 3.3. ■

The following theorem is similar to Theorem 4.3 in [14] for a trilattice logic.

THEOREM 3.5 (Semantical embedding from ML_n into LK)
Let Γ and Δ be sets of formulas in $\mathcal{L}_{\mathrm{ML}}$, and let f be the mapping defined in Definition 2.3.

$$\mathrm{ML}_n \models \Gamma \Rightarrow \Delta \text{ iff } \mathrm{LK} \models f(\Gamma) \Rightarrow f(\Delta).$$

PROOF. ($\Longrightarrow$) Assume that if $v^p(\alpha)=t$ for every $\alpha\in\Gamma$, then $v^p(\beta)=t$ for some $\beta\in\Delta$. Let $v(f(\alpha))=t$ for every $f(\alpha)\in f(\Gamma)$. By Lemma 3.4, we can construct a paraconsistent valuation v^p, such that $v^p(\alpha)=t$ for any $\alpha\in\Gamma$. Thus, $v^p(\beta)=t$ for some $\beta\in\Delta$, and by Lemma 3.3, $v(f(\beta))=t$ for some $f(\beta)\in f(\Delta)$.
($\Longleftarrow$) Similarly. ■

THEOREM 3.6 (Completeness for ML_n)
For any sets Γ and Δ of formulas,

$$\mathrm{ML}_n \vdash \Gamma \Rightarrow \Delta \text{ iff } \mathrm{ML}_n \models \Gamma \Rightarrow \Delta.$$

PROOF. We have the following. $\mathrm{ML}_n \models \Gamma \Rightarrow \Delta$ iff $\mathrm{LK} \models f(\Gamma) \Rightarrow f(\Delta)$ (by Theorem 3.5) iff $\mathrm{LK} \vdash f(\Gamma) \Rightarrow f(\Delta)$ (by the completeness theorem for LK) iff $\mathrm{ML}_n \vdash \Gamma \Rightarrow \Delta$ (by Theorem 2.6). ■

Next, we show a theorem for semantic embedding of LK into ML_n.

LEMMA 3.7
Let g be the mapping defined in Definition 2.8. For any valuation v, we can construct a paraconsistent valuation v^p such that for any formula α,

$$v(\alpha) = t \text{ iff } v^p(g(\alpha)) = t.$$

PROOF. Let Φ be a set of propositional variables and Φ^j $(1 \leq j \leq n)$ be the sets $\{p^j \mid p \in \Phi\}$ of propositional variables. Suppose that v^p is a paraconsistent valuation. Suppose that v is a mapping from $\Phi \cup \bigcup_{1 \leq j \leq n} \Phi^j$ to $\{t, f\}$ such that

(1) $v^p(p) = t$ iff $v(p) = t$,
(2) $v^p({\sim_j} p) = t$ iff $v(p^j) = t$.

Then, the lemma is proved by induction on α.
• Base step:

(1) Case when α is p, where p is a propositional variable: $v(p) = t$ iff $v^p(p) = t$ (by the assumption) iff $v^p(g(p)) = t$ (by the definition of g).
(2) Case when α is p^j, where p is a propositional variable: $v(p^j) = t$ iff $v^p({\sim_j} p) = t$ (by the assumption) iff $v^p(g(p^j)) = t$ (by the definition of g).

• Induction step: We show some cases.

(1) Case when α is $\beta \wedge \gamma$: $v(\beta \wedge \gamma) = t$ iff $v(\beta) = t$ and $v(\gamma) = t$ iff $v^p(g(\beta)) = t$ and $v^p(g(\gamma)) = t$ (by induction hypothesis) iff $v^p(g(\beta) \wedge_j g(\gamma)) = t$ iff $v^p(g(\beta \wedge \gamma)) = t$ (by the definition of g).
(2) Case when α is $\neg\beta$: $v(\neg\beta) = t$ iff $v(\beta) = f$ iff $v^p(g(\beta)) = f$ (by induction hypothesis) iff $v^p({\sim_k}{\sim_j} g(\beta)) = t$ iff $v^p(g(\neg\beta)) = t$ (by the definition of g).

■

LEMMA 3.8
Let g be the mapping defined in Definition 2.8. For any paraconsistent valuation v^p, we can construct a valuation v such that for any formula α,

$$v^p(g(\alpha)) = t \text{ iff } v(\alpha) = t.$$

PROOF. Similar to the proof of Lemma 3.7. ■

THEOREM 3.9 (Semantical embedding from LK into ML_n)
Let Γ and Δ be sets of formulas in $\mathcal{L}_{\mathrm{LK}}$, and let g be the mapping defined in Definition 2.8.

$$\mathrm{LK} \models \Gamma \Rightarrow \Delta \text{ iff } \mathrm{ML}_n \models g(\Gamma) \Rightarrow g(\Delta).$$

PROOF. By Lemmas 3.7 and 3.8. ■

4 Interpolation and separation

In what follows an expression $V(\alpha)$ denotes the set of all propositional variables occurring in α. The following theorem is well known [3, 21]:

THEOREM 4.1 (Craig interpolation for LK)
Suppose LK $\vdash \alpha \Rightarrow \beta$ for any formulas α and β. If $V(\alpha) \cap V(\beta) \neq \emptyset$, then there exists a formula γ such that

(1) LK $\vdash \alpha \Rightarrow \gamma$ and LK $\vdash \gamma \Rightarrow \beta$,
(2) $V(\gamma) \subseteq V(\alpha) \cap V(\beta)$.
If $V(\alpha) \cap V(\beta) = \emptyset$, then
(3) LK $\vdash \ \Rightarrow \neg\alpha$ or LK $\vdash \ \Rightarrow \beta$.

In this section, we establish the modified Craig interpolation theorem for ML$_n$. The next two lemmas are similar to Lemmas 5.2 and 5.3 in [14] for a trilattice logic.

LEMMA 4.2
Let n (>1) be the positive integer determined by n-lattice, let j be any positive integer with $j \leq n$, let I_p be $\{p\} \cup \{p^i \mid 1 \leq i \leq n\}$, and let f be the mapping defined in Definition 2.3. For any propositional variable p in $\mathcal{L}_{\text{ML}}$ and any formula α in $\mathcal{L}_{\text{ML}}$,

(1) $p \in V(\alpha)$ iff $q \in V(f(\alpha))$ for some $q \in I_p$,
(2) $p \in V(\sim_j \alpha)$ iff $q \in V(f(\sim_j \alpha))$ for some $q \in I_p$.

In the following discussion, the subscript of I_p is omitted for the sake of brevity.

PROOF. By (simultaneous) induction on α.
• Base step. For (1), we have: $p \in V(p)$ iff $p = f(p) \in V(f(p))$ by the definition of f. For (2), we have: $p \in V(\sim_j p)$ iff $p^j = f(\sim_j p) \in V(f(\sim_j p))$ by the definition of f.
• Induction step. We show some cases.

(1) Case ($\alpha \equiv \sim_j \beta$). For (1), we obtain:

$p \in V(\sim_j \beta)$
iff $q \in V(f(\sim_j \beta))$ for some $q \in I$ (by induction hypothesis for (2).

For (2), we obtain:

$p \in V(\sim_j \sim_j \beta)$
iff $p \in V(\beta)$
iff $q \in V(f(\beta))$ for some $q \in I$ (by induction hypothesis for (1)
iff $q \in V(f(\sim_j \sim_j \beta))$ for some $q \in I$ (by the definition of f).

(2) Case ($\alpha \equiv \sim_k \beta$ with $k \neq j$). For (1), we obtain the required fact by the same way as Case ($\alpha \equiv \sim_j \beta$). For (2), we obtain:

$p \in V(\sim_j \sim_k \beta)$
iff $p \in V(\beta)$
iff $q \in V(f(\beta))$ for some $q \in I$ (by induction hypothesis for (1)
iff $q \in V(\neg f(\beta))$ for some $q \in I$
iff $q \in V(f(\sim_j \sim_k \beta))$ for some $q \in I$ (by the definition of f).

(3) Case ($\alpha \equiv \beta \wedge_j \gamma$). For (1), we obtain:

$p \in V(\beta \wedge_j \gamma)$
iff $p \in V(\beta)$ or $p \in V(\gamma)$
iff $[r \in V(f(\beta))$ for some $r \in I]$ or $[s \in V(f(\gamma))$ for some $s \in I]$ (by induction hypothesis for (1)
iff $q \in V(f(\beta) \wedge f(\gamma))$ for some $q \in I$
iff $q \in V(f(\beta \wedge_j \gamma))$ for some $q \in I$ (by the definition of f).

For (2), we obtain:

$p \in V(\sim_j(\beta \wedge_j \gamma))$
iff $p \in V(\sim_j \beta)$ or $p \in V(\sim_j \gamma)$
iff $[r \in V(f(\sim_j \beta))$ for some $r \in I]$ or $[s \in V(f(\sim_j \gamma))$ for some $s \in I]$ (by induction hypothesis for (2)
iff $q \in V(f(\sim_j \beta) \vee f(\sim_j \gamma))$ for some $q \in I$
iff $q \in V(f(\sim_j(\beta \wedge_j \gamma)))$ for some $q \in I$ (by the definition of f).

(4) Case ($\alpha \equiv \beta \wedge_k \gamma$ with $k \neq j$). For (1), we obtain the required fact by the same way as Case ($\alpha \equiv \beta \wedge_j \gamma$). For (2), we obtain:

$p \in V(\sim_j(\beta \wedge_k \gamma))$
iff $p \in V(\sim_j \beta)$ or $p \in V(\sim_j \gamma)$
iff $[r \in V(f(\sim_j \beta))$ for some $r \in I]$ or $[s \in V(f(\sim_j \gamma))$ for some $s \in I]$ (by induction hypothesis for (2)
iff $q \in V(f(\sim_j \beta) \wedge f(\sim_j \gamma))$ for some $q \in I$
iff $q \in V(f(\sim_j(\beta \wedge_k \gamma)))$ for some $q \in I$ (by the definition of f).

■

LEMMA 4.3
Let f be the mapping defined in Definition 2.3. For any formulas α and β in $\mathcal{L}_{\mathrm{ML}}$, if $V(f(\alpha)) \subseteq V(f(\beta))$, then $V(\alpha) \subseteq V(\beta)$.

PROOF. Assume $V(f(\alpha)) \subseteq V(f(\beta))$, and let $p \in V(\alpha)$. Then, by Lemma 4.2, we obtain $q \in V(f(\alpha))$ for some $q \in I$. By the assumption, $q \in V(f(\beta))$ for some $q \in I$, and hence, $p \in V(\beta)$ by Lemma 4.2. ■

By using Lemma 4.3, we obtain the following theorem. A similar theorem was shown for a trilattice logic as Theorem 5.4 in [14].

THEOREM 4.4 (Modified Craig interpolation for ML_n)
Let n (>1) be the positive integer determined by n-lattice, and let j, k be any positive integers with $j, k \leq n$ and $j \neq k$. Suppose $\mathrm{ML}_n \vdash \alpha \Rightarrow \beta$ for any formulas α and β. If $V(\alpha) \cap V(\beta) \neq \emptyset$, then there exists a formula γ such that

(1) $\mathrm{ML}_n \vdash \alpha \Rightarrow \gamma$ and $\mathrm{ML}_n \vdash \gamma \Rightarrow \beta$,
(2) $V(\gamma) \subseteq V(\alpha) \cap V(\beta)$.
If $V(\alpha) \cap V(\beta) = \emptyset$, then
(3) $\mathrm{ML}_n \vdash \Rightarrow \sim_k \sim_j \alpha$ or $\mathrm{ML}_n \vdash \Rightarrow \beta$.

PROOF. • Case $V(\alpha) \cap V(\beta) \neq \emptyset$: Suppose $\mathrm{ML}_n \vdash \alpha \Rightarrow \beta$ and $V(\alpha) \cap V(\beta) \neq \emptyset$. Then, we have $\mathrm{LK} \vdash f(\alpha) \Rightarrow f(\beta)$ by Theorem 2.6. By Theorem 4.1, we have the following: there exists a formula γ in

$\mathcal{L}_{\mathrm{LK}}$ such that

(1) $\mathrm{LK} \vdash f(\alpha) \Rightarrow \gamma$ and $\mathrm{LK} \vdash \gamma \Rightarrow f(\beta)$,
(2) $V(\gamma) \subseteq V(f(\alpha)) \cap V(f(\beta))$.

Since γ is a formula of LK, γ is regarded as in $\mathcal{L}^* = \mathcal{L}_{\mathrm{LK}} - \bigcup_{1\leq j\leq n} \Phi^j$ ($\subseteq \mathcal{L}_{\mathrm{ML}}$). Then, we have the fact $\gamma = f(\gamma)$ for any $\gamma \in \mathcal{L}^*$. This fact can be shown by induction on γ. By Theorem 2.6, we thus obtain the following: there exists a formula γ such that

(1) $\mathrm{ML}_n \vdash \alpha \Rightarrow \gamma$ and $\mathrm{ML}_n \vdash \gamma \Rightarrow \beta$,
(2) $V(f(\gamma)) \subseteq V(f(\alpha)) \cap V(f(\beta))$.

Now it is sufficient to show that $V(f(\gamma)) \subseteq V(f(\alpha)) \cap V(f(\beta))$ implies $V(\gamma) \subseteq V(\alpha) \cap V(\beta)$. This is shown by Lemma 4.3.

• Case $V(\alpha) \cap V(\beta) = \emptyset$: Suppose $\mathrm{ML}_n \vdash \alpha \Rightarrow \beta$ and $V(\alpha) \cap V(\beta) = \emptyset$. Then, we have $\mathrm{LK} \vdash f(\alpha) \Rightarrow f(\beta)$ by Theorem 2.6. We also have (*): $V(f(\alpha)) \cap V(f(\beta)) = \emptyset$. To show this, it is sufficient to prove that $V(\alpha) \cap V(\beta) = \emptyset$ implies $V(f(\alpha)) \cap V(f(\beta)) = \emptyset$. We now show the contraposition, i.e. $V(f(\alpha)) \cap V(f(\beta)) \neq \emptyset$ implies $V(\alpha) \cap V(\beta) \neq \emptyset$. Suppose $q \in V(f(\alpha)) \cap V(f(\beta))$ with $q \in \Phi \cup \Phi^1 \cup \cdots \cup \Phi^n$. If q is of the form $f(p) = p \in \Phi$, then we obviously have $p \in V(\alpha) \cap V(\beta)$. If q is of the form $f({\sim_j} p) = p^j \in \Phi^j$, then we have $p \in V(\alpha) \cap V(\beta)$. Therefore, we obtain (*). Thus, by Theorem 4.1, we have the following:

$\mathrm{LK} \vdash \; \Rightarrow \neg f(\alpha)$ or $\mathrm{LK} \vdash \; \Rightarrow f(\beta)$

where $\neg f(\alpha)$ coincides with $f({\sim_k}{\sim_j}\alpha)$ by the definition of f. By Theorem 2.6, we thus obtain the required fact:

$\mathrm{ML}_n \vdash \; \Rightarrow {\sim_k}{\sim_j}\alpha$ or $\mathrm{ML}_n \vdash \; \Rightarrow \beta$.

■

By using Theorem 4.4, we obtain the following theorem. A similar theorem was shown for a trilattice logic as Corollary 5.5 in [14].

THEOREM 4.5 (Maksimova's separation for ML_n)
Let n (>1) be the positive integer determined by n-lattice, and let i be any positive integer with $i \leq n$. Suppose $V(\alpha_1, \alpha_2) \cap V(\beta_1, \beta_2) \neq \emptyset$ for any formulas $\alpha_1, \alpha_2, \beta_1$ and β_2. If $\mathrm{ML}_n \vdash \alpha_1 \wedge_i \beta_1 \Rightarrow \alpha_2 \vee_i \beta_2$, then either $\mathrm{ML}_n \vdash \alpha_1 \Rightarrow \alpha_2$ or $\mathrm{ML}_n \vdash \beta_1 \Rightarrow \beta_2$.

PROOF. Suppose $V(\alpha_1, \alpha_2) \cap V(\beta_1, \beta_2) \neq \emptyset$ and $\mathrm{ML}_n \vdash \alpha_1 \wedge_i \beta_1 \Rightarrow \alpha_2 \vee_i \beta_2$. Then, we have: $\mathrm{ML}_n \vdash \alpha_1, \beta_1 \Rightarrow \alpha_2, \beta_2$,[1] and hence have: $\mathrm{ML}_n \vdash \alpha_1, {\sim_k}{\sim_j}\alpha_2 \Rightarrow {\sim_k}{\sim_j}\beta_1, \beta_2$ with $k \neq j$. Thus, we obtain: $\mathrm{ML}_n \vdash \alpha_1 \wedge_i {\sim_k}{\sim_j}\alpha_2 \Rightarrow {\sim_k}{\sim_j}\beta_1 \vee_i \beta_2$. By Theorem 4.4, we obtain:

$\mathrm{ML}_n \vdash \; \Rightarrow {\sim_k}{\sim_j}(\alpha_1 \wedge_i {\sim_k}{\sim_j}\alpha_2)$ or $\mathrm{ML}_n \vdash \; \Rightarrow {\sim_k}{\sim_j}\beta_1 \vee_i \beta_2$.

We thus obtain the required fact:

$\mathrm{ML}_n \vdash \alpha_1 \Rightarrow \alpha_2$ or $\mathrm{ML}_n \vdash \beta_1 \Rightarrow \beta_2$

[1] Strictly speaking, in order to show this fact, we need to show the invertibility of the logical inference rules concerning $\wedge_i$ and $\vee_i$. The invertibility can straightforwardly be shown.

by:

$$
\dfrac{\Rightarrow\sim_k\sim_j(\alpha_1\wedge_i\sim_k\sim_j\alpha_2) \qquad \dfrac{\dfrac{\dfrac{\dfrac{\vdots}{\alpha_1\Rightarrow\alpha_1}}{\alpha_1\Rightarrow\alpha_2,\alpha_1} \qquad \dfrac{\dfrac{\dfrac{\vdots}{\alpha_2\Rightarrow\alpha_2}}{\alpha_2,\alpha_1\Rightarrow\alpha_2}}{\alpha_1\Rightarrow\alpha_2,\sim_k\sim_j\alpha_2}}{\alpha_1\Rightarrow\alpha_2,\alpha_1\wedge_i\sim_k\sim_j\alpha_2}}{\sim_k\sim_j(\alpha_1\wedge_i\sim_k\sim_j\alpha_2),\alpha_1\Rightarrow\alpha_2}}{\alpha_1\Rightarrow\alpha_2}\ (\text{cut})
$$

or

$$
\dfrac{\Rightarrow\sim_k\sim_j\beta_1\vee_i\beta_2 \qquad \dfrac{\dfrac{\dfrac{\dfrac{\vdots}{\beta_1\Rightarrow\beta_1}}{\beta_1\Rightarrow\beta_1,\beta_2}}{\sim_k\sim_j\beta_1,\beta_1\Rightarrow\beta_2} \qquad \dfrac{\dfrac{\vdots}{\beta_2\Rightarrow\beta_2}}{\beta_2,\beta_1\Rightarrow\beta_2}}{\sim_k\sim_j\beta_1\vee_i\beta_2,\beta_1\Rightarrow\beta_2}}{\beta_1\Rightarrow\beta_2}\ (\text{cut}),
$$

respectively. ■

5 Some extensions

5.1 *Sequent calculi, semantics and translations*

In this section, we extend ML_n to a first-order sequent calculus FML_n with implications and co-implications. The notational conventions of the propositional case are also adopted. To begin with, we introduce the first-order language (without individual constants and function symbols) $\mathcal{L}_{\mathrm{FML}}$. This is denoted simply as $\mathcal{L}$ when no confusion can arise. Thus, *formulas* of $\mathcal{L}$ are standardly constructed from countably many predicate symbols, countably many individual variables, and the logical connectives $\wedge_j$, $\vee_j$, $\sim_j$, $\rightarrow_j$, $\leftarrow_j$, $\forall_j$ and $\exists_j$ for every $j \leq n$. Small letters $p,q,\ldots$ are used to denote predicate symbols, and small letters $x,y,\ldots$ are used to denote individual variables. An expression $\alpha[y/x]$ means the formula which is obtained from the formula α by replacing all free occurrences of the individual variable x in α with the individual variable y, but avoiding a clash of variables by a suitable renaming of bound variables. A 0-ary predicate is regarded as a propositional variable. If Φ is the set of all atomic formulas of $\mathcal{L}$, we then say that $\mathcal{L}$ *is based* on Φ.

Definition 5.1 (FML_n)
Let n (>1) be the positive integer determined by n-lattice, and j,k be any positive integers with $j,k \leq n$ and $j \neq k$.

The sequent calculus FML_n is obtained from ML_n by adding the logical inference rules of the form:

$$
\dfrac{\Gamma\Rightarrow\Delta,\alpha \quad \beta,\Sigma\Rightarrow\Pi}{\alpha\rightarrow_j\beta,\Gamma,\Sigma\Rightarrow\Delta,\Pi}\ (\rightarrow_j\text{left}) \qquad \dfrac{\alpha,\Gamma\Rightarrow\Delta,\beta}{\Gamma\Rightarrow\Delta,\alpha\rightarrow_j\beta}\ (\rightarrow_j\text{right})
$$

$$
\dfrac{\alpha,\Gamma\Rightarrow\Delta,\beta}{\alpha\leftarrow_j\beta,\Gamma\Rightarrow\Delta}\ (\leftarrow_j\text{left}) \qquad \dfrac{\Gamma\Rightarrow\Delta,\alpha \quad \beta,\Sigma\Rightarrow\Pi}{\Gamma,\Sigma\Rightarrow\Delta,\Pi,\alpha\leftarrow_j\beta}\ (\leftarrow_j\text{right})
$$

$$
\dfrac{\alpha[y/x],\Gamma\Rightarrow\Delta}{\forall_j x\alpha,\Gamma\Rightarrow\Delta}\ (\forall_j\text{left}) \qquad \dfrac{\Gamma\Rightarrow\Delta,\alpha[z/x]}{\Gamma\Rightarrow\Delta,\forall_j x\alpha}\ (\forall_j\text{right})
$$

$$\frac{\alpha[z/x], \Gamma \Rightarrow \Delta}{\exists_j x\alpha, \Gamma \Rightarrow \Delta}\ (\exists_j\text{left}) \qquad \frac{\Gamma \Rightarrow \Delta, \alpha[y/x]}{\Gamma \Rightarrow \Delta, \exists_j x\alpha}\ (\exists_j\text{right})$$

$$\frac{\sim_j\beta, \Gamma \Rightarrow \Delta, \sim_j\alpha}{\sim_j(\alpha\rightarrow_j\beta), \Gamma \Rightarrow \Delta}\ (\sim_j\rightarrow_j\text{left}) \qquad \frac{\Gamma \Rightarrow \Delta, \sim_j\beta \quad \sim_j\alpha, \Sigma \Rightarrow \Pi}{\Gamma, \Sigma \Rightarrow \Delta, \Pi, \sim_j(\alpha\rightarrow_j\beta)}\ (\sim_j\rightarrow_j\text{right})$$

$$\frac{\Gamma \Rightarrow \Delta, \sim_j\beta \quad \sim_j\alpha, \Sigma \Rightarrow \Pi}{\sim_j(\alpha\leftarrow_j\beta), \Gamma, \Sigma \Rightarrow \Delta, \Pi}\ (\sim_j\leftarrow_j\text{left}) \qquad \frac{\sim_j\beta, \Gamma \Rightarrow \Delta, \sim_j\alpha}{\Gamma \Rightarrow \Delta, \sim_j(\alpha\leftarrow_j\beta)}\ (\sim_j\leftarrow_j\text{right})$$

$$\frac{\sim_j\alpha[z/x], \Gamma \Rightarrow \Delta}{\sim_j\forall_j x\alpha, \Gamma \Rightarrow \Delta}\ (\sim_j\forall_j\text{left}) \qquad \frac{\Gamma \Rightarrow \Delta, \sim_j\alpha[y/x]}{\Gamma \Rightarrow \Delta, \sim_j\forall_j x\alpha}\ (\sim_j\forall_j\text{right})$$

$$\frac{\sim_j\alpha[y/x], \Gamma \Rightarrow \Delta}{\sim_j\exists_j x\alpha, \Gamma \Rightarrow \Delta}\ (\sim_j\exists_j\text{left}) \qquad \frac{\Gamma \Rightarrow \Delta, \sim_j\alpha[z/x]}{\Gamma \Rightarrow \Delta, \sim_j\exists_j x\alpha}\ (\sim_j\exists_j\text{right})$$

$$\frac{\Gamma \Rightarrow \Delta, \sim_k\alpha \quad \sim_k\beta, \Sigma \Rightarrow \Pi}{\sim_k(\alpha\rightarrow_j\beta), \Gamma, \Sigma \Rightarrow \Delta, \Pi}\ (\sim_k\rightarrow_j\text{left}) \qquad \frac{\sim_k\alpha, \Gamma \Rightarrow \Delta, \sim_k\beta}{\Gamma \Rightarrow \Delta, \sim_k(\alpha\rightarrow_j\beta)}\ (\sim_k\rightarrow_j\text{right})$$

$$\frac{\sim_k\alpha, \Gamma \Rightarrow \Delta, \sim_k\beta}{\sim_k(\alpha\leftarrow_j\beta), \Gamma \Rightarrow \Delta}\ (\sim_k\leftarrow_j\text{left}) \qquad \frac{\Gamma \Rightarrow \Delta, \sim_k\alpha \quad \sim_k\beta, \Sigma \Rightarrow \Pi}{\Gamma, \Sigma \Rightarrow \Delta, \Pi, \sim_k(\alpha\leftarrow_j\beta)}\ (\sim_k\leftarrow_j\text{right})$$

$$\frac{\sim_k\alpha[y/x], \Gamma \Rightarrow \Delta}{\sim_k\forall_j x\alpha, \Gamma \Rightarrow \Delta}\ (\sim_k\forall_j\text{left}) \qquad \frac{\Gamma \Rightarrow \Delta, \sim_k\alpha[z/x]}{\Gamma \Rightarrow \Delta, \sim_k\forall_j x\alpha}\ (\sim_k\forall_j\text{right})$$

$$\frac{\sim_k\alpha[z/x], \Gamma \Rightarrow \Delta}{\sim_k\exists_j x\alpha, \Gamma \Rightarrow \Delta}\ (\sim_k\exists_j\text{left}) \qquad \frac{\Gamma \Rightarrow \Delta, \sim_j\alpha[y/x]}{\Gamma \Rightarrow \Delta, \sim_k\exists_j x\alpha}\ (\sim_k\exists_j\text{right})$$

where y is an arbitrary individual variable, and z is an individual variable which has the eigenvariable condition, i.e. z does not occur as a free individual variable in the lower sequent of the rule. It is remarked that a propositional variable p in the initial sequent definition of ML_n is replaced by an atomic formula p' in the initial sequent definition of FML_n.

Some remarks are given as follows.

(1) An expression $\alpha \Leftrightarrow \beta$ represents the sequents $\alpha \Rightarrow \beta$ and $\beta \Rightarrow \alpha$. The following sequents are provable in cut-free FML_n:

  (a) $\sim_j(\alpha\rightarrow_j\beta) \Leftrightarrow \sim_j\beta \leftarrow_j \sim_j\alpha$,
  (b) $\sim_j(\alpha\leftarrow_j\beta) \Leftrightarrow \sim_j\beta \rightarrow_j \sim_j\alpha$,
  (c) $\sim_k(\alpha\rightarrow_j\beta) \Leftrightarrow \sim_k\alpha \rightarrow_j \sim_k\beta$,
  (d) $\sim_k(\alpha\leftarrow_j\beta) \Leftrightarrow \sim_k\alpha \leftarrow_j \sim_k\beta$.

(2) Some sequent calculi that can prove the following sequents, which are similar to the sequents presented above, were studied in [15, 23].

  (a) $\sim(\alpha\rightarrow\beta) \Leftrightarrow \sim\beta \leftarrow \sim\alpha$ (negated implication as contraposed co-implication),
  (b) $\sim(\alpha\leftarrow\beta) \Leftrightarrow \sim\beta \rightarrow \sim\alpha$ (negated co-implication as contraposed implication).

A Gentzen-type sequent calculus FLK for first-order classical logic with implication and co-implication is introduced below. The first-order language $\mathcal{L}_{\mathrm{FLK}}$ is obtained from $\mathcal{L}_{\mathrm{FML}}$ by deleting $\sim_j$, adding $\neg$ and replacing $\wedge_j$, $\vee_j$, $\rightarrow_j$, $\leftarrow_j$, $\forall_j$ and $\exists_j$ with $\wedge$, $\vee$, $\rightarrow$, $\leftarrow$, $\forall$ and $\exists$. This language will also be denoted simply by $\mathcal{L}$ when no confusion can arise.

DEFINITION 5.2 (FLK)
The sequent calculus FLK is obtained from LK by adding the logical inference rules of the form:

$$\frac{\Gamma \Rightarrow \Delta, \alpha \quad \beta, \Sigma \Rightarrow \Pi}{\alpha \to \beta, \Gamma, \Sigma \Rightarrow \Delta, \Pi}\ (\to\text{left}) \qquad \frac{\alpha, \Gamma \Rightarrow \Delta, \beta}{\Gamma \Rightarrow \Delta, \alpha \to \beta}\ (\to\text{right})$$

$$\frac{\alpha, \Gamma \Rightarrow \Delta, \beta}{\alpha \leftarrow \beta, \Gamma \Rightarrow \Delta}\ (\leftarrow\text{left}) \qquad \frac{\Gamma \Rightarrow \Delta, \alpha \quad \beta, \Sigma \Rightarrow \Pi}{\Gamma, \Sigma \Rightarrow \Delta, \Pi, \alpha \leftarrow \beta}\ (\leftarrow\text{right})$$

$$\frac{\alpha[y/x], \Gamma \Rightarrow \Delta}{\forall x \alpha, \Gamma \Rightarrow \Delta}\ (\forall\text{left}) \qquad \frac{\Gamma \Rightarrow \Delta, \alpha[z/x]}{\Gamma \Rightarrow \Delta, \forall x \alpha}\ (\forall\text{right})$$

$$\frac{\alpha[z/x], \Gamma \Rightarrow \Delta}{\exists x \alpha, \Gamma \Rightarrow \Delta}\ (\exists\text{left}) \qquad \frac{\Gamma \Rightarrow \Delta, \alpha[y/x]}{\Gamma \Rightarrow \Delta, \exists x \alpha}\ (\exists\text{right}),$$

where y is an arbitrary individual variable, and z is an individual variable which has the eigenvariable condition. It is remarked that a propositional variable p in the initial sequent definition of LK is replaced by an atomic formula p' in the initial sequent definition of FLK.

Some remarks are given as follows.

(1) FLK is equivalent to (the first-order version of) Crolard's sequent calculus SLK for (classical) *subtractive logic* [4].
(2) The co-implication connective $\leftarrow$, which is characterized by the inference rules ($\leftarrow$left) and ($\leftarrow$right), is definable in Gentzen's original LK.
(3) By the above-mentioned facts, FLK, SLK and Gentzen's original LK are logically equivalent.
(4) FLK is thus not a new logic, but just an alternative formalization of classical logic.
(5) A paraconsistent extension of FLK, called a *symmetric paraconsistent logic*, was studied in [15].
(6) The cut-elimination theorem holds for FLK.

Next, semantics for FML$_n$ and FLK are introduced below.

DEFINITION 5.3
Let n (>1) be the positive integer determined by n-lattice, and j,k be any positive integers with $j,k \leq n$ and $j \neq k$.

A structure $\mathcal{A} := \langle U, I^p \rangle$ is called a *paraconsistent model* if the following conditions hold:

(1) U is a non-empty set,
(2) I^p and $(\sim_j p)^{I^p}$ are mappings such that $p^{I^p}, (\sim_j p)^{I^p} \subseteq U^n$ (i.e. p^{I^p} and $(\sim_j p)^{I^p}$ are n-ary relations on U) for an n-ary predicate symbol p.

We introduce the notation $\underline{u}$ as the name of $u \in U$, and we denote as $\mathcal{L}[\mathcal{A}]$ the language obtained from $\mathcal{L}$ by adding the names of all the elements of U. A formula α is called a *closed formula* if α has no free individual variable. A formula of the form $\forall_j x_1 \cdots \forall_j x_m \alpha$ is called the *universal closure* of α if the free variables of α are $x_1, \ldots, x_m$. We write $cl(\alpha)$ for the universal closure of α.

DEFINITION 5.4 (Semantics for FML$_n$)
Let $\mathcal{A} := \langle U, I^p \rangle$ be a paraconsistent model, and j,k be any positive integers with $j,k \leq n$ and $j \neq k$. The paraconsistent satisfaction relation $\mathcal{A} \models^p \alpha$ for any closed formula α of $\mathcal{L}[\mathcal{A}]$ are defined inductively by:

(1) $[\mathcal{A} \models^p p(\underline{x}_1, \ldots, \underline{x}_n)$ iff $(x_1, \ldots, x_n) \in p^{I^p}]$ for any n-ary atomic formula $p(\underline{x}_1, \ldots, \underline{x}_n)$,

(2) $[\mathcal{A} \models^p \sim_j p(\underline{x}_1,...,\underline{x}_n)$ iff $(x_1,...,x_n) \in (\sim_j p)^{I^p}]$ for any n-ary negated atomic formula $\sim_j p(\underline{x}_1,...,\underline{x}_n)$,
(3) $\mathcal{A} \models^p \alpha \wedge_j \beta$ iff $\mathcal{A} \models^p \alpha$ and $\mathcal{A} \models^p \beta$,
(4) $\mathcal{A} \models^p \alpha \vee_j \beta$ iff $\mathcal{A} \models^p \alpha$ or $\mathcal{A} \models^p \beta$,
(5) $\mathcal{A} \models^p \alpha \rightarrow_j \beta$ iff $\mathcal{A} \not\models^p \alpha$ or $\mathcal{A} \models^p \beta$,
(6) $\mathcal{A} \models^p \alpha \leftarrow_j \beta$ iff $\mathcal{A} \models^p \alpha$ and $\mathcal{A} \not\models^p \beta$,
(7) $\mathcal{A} \models^p \forall_j x\alpha$ iff $\mathcal{A} \models^p \alpha[\underline{u}/x]$ for all $u \in U$,
(8) $\mathcal{A} \models^p \exists_j x\alpha$ iff $\mathcal{A} \models^p \alpha[\underline{u}/x]$ for some $u \in U$,
(9) $\mathcal{A} \models^p \sim_j \sim_j \alpha$ iff $\mathcal{A} \models^p \alpha$,
(10) $\mathcal{A} \models^p \sim_j(\alpha \wedge_j \beta)$ iff $\mathcal{A} \models^p \sim_j \alpha$ or $\mathcal{A} \models^p \sim_j \beta$,
(11) $\mathcal{A} \models^p \sim_j(\alpha \vee_j \beta)$ iff $\mathcal{A} \models^p \sim_j \alpha$ and $\mathcal{A} \models^p \sim_j \beta$,
(12) $\mathcal{A} \models^p \sim_j(\alpha \rightarrow_j \beta)$ iff $\mathcal{A} \models^p \sim_j \beta$ and $\mathcal{A} \not\models^p \sim_j \alpha$,
(13) $\mathcal{A} \models^p \sim_j(\alpha \leftarrow_j \beta)$ iff $\mathcal{A} \not\models^p \sim_j \beta$ or $\mathcal{A} \models^p \sim_j \alpha$,
(14) $\mathcal{A} \models^p \sim_j \forall_j x\alpha$ iff $\mathcal{A} \models^p \sim_j \alpha[\underline{u}/x]$ for some $u \in U$,
(15) $\mathcal{A} \models^p \sim_j \exists_j x\alpha$ iff $\mathcal{A} \models^p \sim_j \alpha[\underline{u}/x]$ for all $u \in U$,
(16) $\mathcal{A} \models^p \sim_k \sim_j \alpha$ iff $\mathcal{A} \not\models^p \alpha$,
(17) $\mathcal{A} \models^p \sim_k(\alpha \wedge_j \beta)$ iff $\mathcal{A} \models^p \sim_k \alpha$ and $\mathcal{A} \models^p \sim_k \beta$,
(18) $\mathcal{A} \models^p \sim_k(\alpha \vee_j \beta)$ iff $\mathcal{A} \models^p \sim_k \alpha$ or $\mathcal{A} \models^p \sim_k \beta$,
(19) $\mathcal{A} \models^p \sim_k(\alpha \rightarrow_j \beta)$ iff $\mathcal{A} \not\models^p \sim_k \alpha$ or $\mathcal{A} \models^p \sim_k \beta$,
(20) $\mathcal{A} \models^p \sim_k(\alpha \leftarrow_j \beta)$ iff $\mathcal{A} \models^p \sim_k \alpha$ and $\mathcal{A} \not\models^p \sim_k \beta$,
(21) $\mathcal{A} \models^p \sim_k \forall_j x\alpha$ iff $\mathcal{A} \models^p \sim_k \alpha[\underline{u}/x]$ for all $u \in U$,
(22) $\mathcal{A} \models^p \sim_k \exists_j x\alpha$ iff $\mathcal{A} \models^p \sim_k \alpha[\underline{u}/x]$ for some $u \in U$.

The paraconsistent satisfaction relation $\mathcal{A} \models^p \alpha$ for any formula α of $\mathcal{L}$ are defined by ($\mathcal{A} \models^p \alpha$ iff $\mathcal{A} \models^p cl(\alpha)$). A formula α of $\mathcal{L}$ is called FML-*valid* iff $\mathcal{A} \models^p \alpha$ holds for any paraconsistent model $\mathcal{A}$. A sequent $\Gamma \Rightarrow \Delta$ of $\mathcal{L}$ is called FML-valid (denoted by $\text{FML}_n \models \Gamma \Rightarrow \Delta$) iff for any paraconsistent model $\mathcal{A}$, if $\mathcal{A} \models^p \gamma$ for all $\gamma \in \Gamma$, then $\mathcal{A} \models^p \delta$ for some $\delta \in \Delta$.

DEFINITION 5.5 (Semantics for FLK)
A structure $\mathcal{A} := \langle U, I \rangle$, called a *model*, is defined in a similar way as in Definition 5.3. Then, the satisfaction relation $\mathcal{A} \models \alpha$ for any closed formula α of $\mathcal{L}[\mathcal{A}]$ is defined inductively by:

(1) $[\mathcal{A} \models p(\underline{x}_1,...,\underline{x}_n)$ iff $(x_1,...,x_n) \in p^I]$ for any n-ary atomic formula $p(\underline{x}_1,...,\underline{x}_n)$,
(2) $\mathcal{A} \models \alpha \wedge \beta$ iff $\mathcal{A} \models \alpha$ and $\mathcal{A} \models \beta$,
(3) $\mathcal{A} \models \alpha \vee \beta$ iff $\mathcal{A} \models \alpha$ or $\mathcal{A} \models \beta$,
(4) $\mathcal{A} \models \alpha \rightarrow \beta$ iff $\mathcal{A} \not\models \alpha$ or $\mathcal{A} \models \beta$,
(5) $\mathcal{A} \models \alpha \leftarrow \beta$ iff $\mathcal{A} \models \alpha$ and $\mathcal{A} \not\models \beta$,
(6) $\mathcal{A} \models \forall x\alpha$ iff $\mathcal{A} \models \alpha[\underline{u}/x]$ for all $u \in U$,
(7) $\mathcal{A} \models \exists x\alpha$ iff $\mathcal{A} \models \alpha[\underline{u}/x]$ for some $u \in U$,
(8) $\mathcal{A} \models \neg\alpha$ iff $\mathcal{A} \not\models \alpha$.

The satisfaction relation $\mathcal{A} \models \alpha$ for any formula α of $\mathcal{L}$ is defined in a similar way as in Definition 5.4. The notions of FLK-validities for a formula and a sequent are also defined in a similar way as in Definition 5.4.

DEFINITION 5.6
Let n (>1) be the positive integer determined by n-lattice, and let j,k be any positive integers with $j,k \leq n$ and $j \neq k$. We fix a set Φ of atomic formulas and define for each ($j \leq n$) the set $\Phi^j := \{p^j \mid p \in \Phi\}$

of atomic formulas. Let languages $\mathcal{L}_{\text{FML}}$ and $\mathcal{L}_{\text{FLK}}$ defined as above, be based on sets Φ and $\Phi \cup \bigcup_j \Phi^j$, respectively.

A mapping f from $\mathcal{L}_{\text{FML}}$ to $\mathcal{L}_{\text{FLK}}$ is obtained then by adding the following conditions to the conditions of the mapping f in Definition 2.3:

(1) $f(\alpha \rightarrow_j \beta) := f(\alpha) \rightarrow f(\beta)$,
(2) $f(\alpha \leftarrow_j \beta) := f(\alpha) \leftarrow f(\beta)$,
(3) $f(\forall_j x\alpha) := \forall x f(\alpha)$,
(4) $f(\exists_j x\alpha) := \exists x f(\alpha)$,
(5) $f(\sim_j(\alpha \rightarrow_j \beta)) := f(\sim_j \beta) \leftarrow f(\sim_j \alpha)$,
(6) $f(\sim_j(\alpha \leftarrow_j \beta)) := f(\sim_j \beta) \rightarrow f(\sim_j \alpha)$,
(7) $f(\sim_j \forall_j x\alpha) := \exists x f(\sim_j \alpha)$,
(8) $f(\sim_j \exists_j x\alpha) := \forall x f(\sim_j \alpha)$,
(9) $f(\sim_k(\alpha \rightarrow_j \beta)) := f(\sim_k \alpha) \rightarrow f(\sim_k \beta)$,
(10) $f(\sim_k(\alpha \leftarrow_j \beta)) := f(\sim_k \alpha) \leftarrow f(\sim_k \beta)$,
(11) $f(\sim_k \forall_j x\alpha) := \forall x f(\sim_k \alpha)$,
(12) $f(\sim_k \exists_j x\alpha) := \exists x f(\sim_k \alpha)$.

In Definition 5.6, we need to remark the differences between the propositional setting and the predicate setting. The expressions p and p^j include, e.g. $p(x_1,x_2)$ and $p^j(x_1,x_2)$, respectively, with some individual variables x_1 and x_2. Also, the expression $f(\sim_j p(x_1,x_2))$ with the mapping f coincides with $p^j(x_1,x_2)$.

Definition 5.7
Let $\mathcal{L}_{\text{FML}}$ and $\mathcal{L}_{\text{FLK}}$ be the languages defined in Definition 5.6. A mapping g from $\mathcal{L}_{\text{FLK}}$ to $\mathcal{L}_{\text{FML}}$ is obtained then by adding the following conditions to the conditions of the mapping g in Definition 2.8:

(1) $g(\alpha \rightarrow \beta) := g(\alpha) \rightarrow_j g(\beta)$, where j is a fixed positive integer, such that $j \leq n$;
(2) $g(\alpha \leftarrow \beta) := g(\alpha) \leftarrow_j g(\beta)$, where j is a fixed positive integer, such that $j \leq n$;
(3) $g(\forall x\alpha) := \forall_j x g(\alpha)$, where j is a fixed positive integer, such that $j \leq n$;
(4) $g(\exists x\alpha) := \exists_j x g(\alpha)$, where j is a fixed positive integer, such that $j \leq n$.

5.2 *Theorems and proofs*

In what follows $P(\alpha)$ denotes the set of all predicate symbols occurring in the formula α. The following theorem extends then to FML_n the corresponding results obtained above for ML_n.

Theorem 5.8
Let n (>1) be the positive integer determined by n-lattice, and let j,k be any positive integers with $j,k \leq n$ and $j \neq k$.

(1) (Syntactical embedding from FML_n into FLK): Let Γ, Δ be sets of formulas in $\mathcal{L}_{\text{FML}}$, and f be the mapping defined in Definition 5.6.

 (a) $\text{FML}_n \vdash \Gamma \Rightarrow \Delta$ iff $\text{FLK} \vdash f(\Gamma) \Rightarrow f(\Delta)$.
 (b) $\text{FML}_n - (\text{cut}) \vdash \Gamma \Rightarrow \Delta$ iff $\text{FLK} - (\text{cut}) \vdash f(\Gamma) \Rightarrow f(\Delta)$.

(2) (Syntactical embedding from FLK into FML_n): Let Γ, Δ be sets of formulas in $\mathcal{L}_{\text{FLK}}$, and g be the mapping defined in Definition 5.7.

(a) FLK $\vdash \Gamma \Rightarrow \Delta$ iff FML$_n \vdash g(\Gamma) \Rightarrow g(\Delta)$.
(b) FLK $-$ (cut) $\vdash \Gamma \Rightarrow \Delta$ iff FML$_n$ $-$ (cut) $\vdash g(\Gamma) \Rightarrow g(\Delta)$.

(3) (Cut-elimination for FML$_n$): The rule (cut) is admissible in cut-free FML$_n$.

(4) (Semantical embedding from FML$_n$ into FLK): Let Γ and Δ be sets of formulas in L$_{\mathrm{FML}}$, and let f be the mapping defined in Definition 5.6.

$$\mathrm{FML}_n \models \Gamma \Rightarrow \Delta \text{ iff } \mathrm{FLK} \models f(\Gamma) \Rightarrow f(\Delta).$$

(5) (Semantical embedding from FLK into FML$_n$): Let Γ and Δ be sets of formulas in L$_{\mathrm{FLK}}$, and let g be the mapping defined in Definition 5.7.

$$\mathrm{FLK} \models \Gamma \Rightarrow \Delta \text{ iff } \mathrm{FML}_n \models g(\Gamma) \Rightarrow g(\Delta).$$

(6) (Completeness for FML$_n$): For any sets Γ and Δ of formulas,

$$\mathrm{FML}_n \vdash \Gamma \Rightarrow \Delta \text{ iff } \mathrm{FML}_n \models \Gamma \Rightarrow \Delta.$$

(7) (Modified Craig interpolation for FML$_n$): Suppose FML$_n \vdash \alpha \Rightarrow \beta$ for any formulas α and β. If $P(\alpha) \cap P(\beta) \neq \emptyset$, then there exists a formula γ such that

(a) FML$_n \vdash \alpha \Rightarrow \gamma$ and FML$_n \vdash \gamma \Rightarrow \beta$,
(b) $V(\gamma) \subseteq P(\alpha) \cap P(\beta)$.[2]
If $P(\alpha) \cap P(\beta) = \emptyset$, then
(c) FML$_n \vdash \Rightarrow \sim_k \sim_j \alpha$ or FML$_n \vdash \Rightarrow \beta$.

(8) (Maksimova's separation for FML$_n$): Suppose $V(\alpha_1, \alpha_2) \cap V(\beta_1, \beta_2) \neq \emptyset$ for any formulas $\alpha_1, \alpha_2, \beta_1$ and β_2. If FML$_n \vdash \alpha_1 \wedge_i \beta_1 \Rightarrow \alpha_2 \vee_i \beta_2$, then either FML$_n \vdash \alpha_1 \Rightarrow \alpha_2$ or FML$_n \vdash \beta_1 \Rightarrow \beta_2$.

The results presented in Theorem 5.8 hold also for the $\{\rightarrow_j, \leftarrow_j\}$-free fragment of FML$_n$ and the $\{\forall_j, \exists_j\}$-free fragment of FML$_n$.

Below we will only demonstrate the syntactical and semantical embedding of FML$_n$ into FLK, for which purpose we will show (i) the weak syntactical embedding theorem of FML$_n$ into FLK and (ii) the key lemma for semantical embedding theorem of FML$_n$ into FLK. The proofs of other results listed in Theorem 5.8 can be obtained in a similar way as for ML$_n$, and thus, they are left to an interested reader as an exercise.

THEOREM 5.9 (Weak syntactical embedding from FML$_n$ into FLK)
Let Γ, Δ be sets of formulas in $\mathcal{L}_{\mathrm{FML}}$, and f be the mapping defined in Definition 5.6.

(1) If FML$_n \vdash \Gamma \Rightarrow \Delta$, then FLK $\vdash f(\Gamma) \Rightarrow f(\Delta)$.
(2) If FLK $-$ (cut) $\vdash f(\Gamma) \Rightarrow f(\Delta)$, then FML$_n$ $-$ (cut) $\vdash \Gamma \Rightarrow \Delta$.

PROOF. • (1) By induction on the proofs P of $\Gamma \Rightarrow \Delta$ in FML$_n$. We distinguish the cases according to the last inference of P, and show some cases.

(1) Case ($\sim_j \rightarrow_j$left): The last inference of P is of the form:

$$\frac{\sim_j \beta, \Gamma \Rightarrow \Delta, \sim_j \alpha}{\sim_j(\alpha \rightarrow_j \beta), \Gamma \Rightarrow \Delta}\ (\sim_j \rightarrow_j \text{left}).$$

[2] Strictly speaking, γ also contains only individual variables that occur in both α and β.

By induction hypothesis, we have FLK $\vdash f(\sim_j\beta), f(\Gamma) \Rightarrow f(\Delta), f(\sim_j\alpha)$. Then, we obtain the required fact:

$$\dfrac{\begin{array}{c}\vdots\\ f(\sim_j\beta), f(\Gamma) \Rightarrow f(\Delta), f(\sim_j\alpha)\end{array}}{f(\sim_j\beta) \leftarrow f(\sim_j\alpha), f(\Gamma) \Rightarrow f(\Delta)}\ (\leftarrow\text{left})$$

where $f(\sim_j\beta) \leftarrow f(\sim_j\alpha)$ coincides with $f(\sim_j(\alpha \rightarrow_j \beta))$ by the definition of f.

(2) Case ($\sim_j\rightarrow_j$right): The last inference of P is of the form:

$$\dfrac{\Gamma \Rightarrow \Delta, \sim_j\beta \quad \sim_j\alpha, \Sigma \Rightarrow \Pi}{\Gamma, \Sigma \Rightarrow \Delta, \Pi, \sim_j(\alpha \rightarrow_j \beta)}\ (\sim_j\rightarrow_j\text{right}).$$

By induction hypothesis, we have FLK $\vdash f(\Gamma) \Rightarrow f(\Delta), f(\sim_j\beta)$ and FLK $\vdash f(\sim_j\alpha), f(\Sigma) \Rightarrow f(\Pi)$. Then, we obtain the required fact:

$$\dfrac{\begin{array}{c}\vdots\\ f(\Gamma) \Rightarrow f(\Delta), f(\sim_j\beta)\end{array} \quad \begin{array}{c}\vdots\\ f(\sim_j\alpha), f(\Sigma) \Rightarrow f(\Pi)\end{array}}{f(\Gamma), f(\Sigma) \Rightarrow f(\Delta), f(\Pi), f(\sim_j\beta) \leftarrow f(\sim_j\alpha)}\ (\leftarrow\text{right}).$$

where $f(\sim_j\beta) \leftarrow f(\sim_j\alpha)$ coincides with $f(\sim_j(\alpha \rightarrow_j \beta))$ by the definition of f.

(3) Case ($\sim_k\rightarrow_j$left): The last inference of P is of the form:

$$\dfrac{\Gamma \Rightarrow \Delta, \sim_k\alpha \quad \sim_k\beta, \Sigma \Rightarrow \Pi}{\sim_k(\alpha \rightarrow_j \beta), \Gamma, \Sigma \Rightarrow \Delta, \Pi}\ (\sim_k\rightarrow_j\text{left}).$$

By induction hypothesis, we have FLK $\vdash f(\Gamma) \Rightarrow f(\Delta), f(\sim_k\alpha)$ and FLK $\vdash f(\sim_k\beta), f(\Sigma) \Rightarrow f(\Pi)$. Then, we obtain the required fact:

$$\dfrac{\begin{array}{c}\vdots\\ f(\Gamma) \Rightarrow f(\Delta), f(\sim_k\alpha)\end{array} \quad \begin{array}{c}\vdots\\ f(\sim_k\beta), f(\Sigma) \Rightarrow f(\Pi)\end{array}}{f(\sim_k\alpha) \rightarrow f(\sim_k\beta), f(\Gamma), f(\Sigma) \Rightarrow f(\Delta), f(\Pi)}\ (\rightarrow\text{left})$$

where $f(\sim_k\alpha) \rightarrow f(\sim_k\beta)$ coincides with $f(\sim_k(\alpha \rightarrow_j \beta))$ by the definition of f.

(4) Case ($\sim_k\rightarrow_j$right): The last inference of P is of the form:

$$\dfrac{\sim_k\alpha, \Gamma \Rightarrow \Delta, \sim_k\beta}{\Gamma \Rightarrow \Delta, \sim_k(\alpha \rightarrow_j \beta)}\ (\sim_k\rightarrow_j\text{right}).$$

By induction hypothesis, we have FLK $\vdash f(\sim_k\alpha), f(\Gamma) \Rightarrow f(\Delta), f(\sim_k\beta)$. Then, we obtain the required fact:

$$\dfrac{\begin{array}{c}\vdots\\ f(\sim_k\alpha), f(\Gamma) \Rightarrow f(\Delta), f(\sim_k\beta)\end{array}}{f(\Gamma) \Rightarrow f(\Delta), f(\sim_k\alpha) \rightarrow f(\sim_k\beta)}\ (\rightarrow\text{right})$$

where $f(\sim_k\alpha) \rightarrow f(\sim_k\beta)$ coincides with $f(\sim_k(\alpha \rightarrow_j \beta))$ by the definition of f.

(5) Case ($\sim_j\forall_j$left): The last inference of P is of the form:

$$\frac{\sim_j\alpha[z/x], \Gamma \Rightarrow \Delta}{\sim_j\forall_j x\alpha, \Gamma \Rightarrow \Delta}\ (\sim_j\forall_j\text{left}).$$

By induction hypothesis, we have FLK $\vdash f(\sim_j\alpha[z/x]), f(\Gamma) \Rightarrow f(\Delta)$. Then, we obtain the required fact:

$$\frac{\begin{array}{c}\vdots\\ f(\sim_j\alpha[z/x]), f(\Gamma) \Rightarrow f(\Delta)\end{array}}{\exists x f(\sim_j\alpha), f(\Gamma) \Rightarrow f(\Delta)}\ (\exists\text{left})$$

where $\exists x f(\sim_j\alpha)$ coincides with $f(\sim_j\forall_j x\alpha)$ by the definition of f.

(6) Case ($\sim_j\forall_j$right): The last inference of P is of the form:

$$\frac{\Gamma \Rightarrow \Delta, \sim_j\alpha[y/x]}{\Gamma \Rightarrow \Delta, \sim_j\forall_j x\alpha}\ (\sim_j\forall_j\text{right}).$$

By induction hypothesis, we have FLK $\vdash f(\Gamma) \Rightarrow f(\Delta), f(\sim_j\alpha[y/x])$. Then, we obtain the required fact:

$$\frac{\begin{array}{c}\vdots\\ f(\Gamma) \Rightarrow f(\Delta), f(\sim_j\alpha[y/x])\end{array}}{f(\Gamma) \Rightarrow f(\Delta), \exists x f(\sim_j\alpha)}\ (\exists\text{right})$$

where $\exists x f(\sim_j\alpha)$ coincides with $f(\sim_j\forall_j x\alpha)$ by the definition of f.

• (2) By induction on the proofs Q of $f(\Gamma) \Rightarrow f(\Delta)$ in FLK $-$ (cut). We distinguish the cases according to the last inference of Q, and show only the following case.

Case ($\rightarrow$left): The last inference of Q is ($\rightarrow$left).

(1) Subcase (1): The last inference of Q is of the form:

$$\frac{f(\Gamma) \Rightarrow f(\Delta), f(\alpha) \quad f(\beta), f(\Sigma) \Rightarrow f(\Pi)}{f(\alpha \rightarrow_j \beta), f(\Gamma), f(\Sigma) \Rightarrow f(\Delta), f(\Pi)}\ (\rightarrow\text{left})$$

where $f(\alpha \rightarrow_j \beta)$ coincides with $f(\alpha) \rightarrow f(\beta)$ by the definition of f. By induction hypothesis, we have FML_n $-$ (cut) $\vdash \Gamma \Rightarrow \Delta, \alpha$ and FML_n $-$ (cut) $\vdash \beta, \Sigma \Rightarrow \Pi$. We thus obtain the required fact:

$$\frac{\begin{array}{c}\vdots\\ \Gamma \Rightarrow \Delta, \alpha\end{array} \quad \begin{array}{c}\vdots\\ \beta, \Sigma \Rightarrow \Pi\end{array}}{\alpha \rightarrow_j \beta, \Gamma, \Sigma \Rightarrow \Delta, \Pi}\ (\rightarrow_j\text{left}).$$

(2) Subcase (2): The last inference of Q is of the form:

$$\frac{f(\Gamma) \Rightarrow f(\Delta), f(\sim_j\alpha) \quad f(\sim_j\beta), f(\Sigma) \Rightarrow f(\Pi)}{f(\sim_j(\beta \leftarrow_j \alpha)), f(\Gamma), f(\Sigma) \Rightarrow f(\Delta), f(\Pi)}\ (\rightarrow\text{left})$$

where $f(\sim_j(\beta \leftarrow_j \alpha))$ coincides with $f(\sim_j\alpha) \rightarrow f(\sim_j\beta)$ by the definition of f. By induction hypothesis, we have FML_n $-$ (cut) $\vdash \Gamma \Rightarrow \Delta, \sim_j\alpha$ and FML_n $-$ (cut) $\vdash \sim_j\beta, \Sigma \Rightarrow \Pi$. We thus obtain the required fact:

$$\dfrac{\begin{array}{cc} \vdots & \vdots \\ \Gamma \Rightarrow \Delta, \sim_j\alpha & \sim_j\beta, \Sigma \Rightarrow \Pi \end{array}}{\sim_j(\beta \leftarrow_j \alpha), \Gamma, \Sigma \Rightarrow \Delta, \Pi}\ (\sim_j\leftarrow_j\text{left}).$$

(3) Subcase (3): The last inference of Q is of the form:

$$\dfrac{f(\Gamma) \Rightarrow f(\Delta), f(\sim_k\alpha) \quad f(\sim_k\beta), f(\Sigma) \Rightarrow f(\Pi)}{f(\sim_k(\alpha \rightarrow_j \beta)), f(\Gamma), f(\Sigma) \Rightarrow f(\Delta), f(\Pi)}\ (\rightarrow\text{left})$$

where $f(\sim_k(\alpha\rightarrow_j\beta))$ coincides with $f(\sim_k\alpha)\rightarrow f(\sim_k\beta)$ by the definition of f. By induction hypothesis, we have $\mathrm{FML}_n - (\mathrm{cut}) \vdash \Gamma \Rightarrow \Delta, \sim_k\alpha$ and $\mathrm{FML}_n - (\mathrm{cut}) \vdash \sim_k\beta, \Sigma \Rightarrow \Pi$. We thus obtain the required fact:

$$\dfrac{\begin{array}{cc} \vdots & \vdots \\ \Gamma \Rightarrow \Delta, \sim_k\alpha & \sim_k\beta, \Sigma \Rightarrow \Pi \end{array}}{\sim_k(\alpha \rightarrow_j \beta), \Gamma, \Sigma \Rightarrow \Delta, \Pi}\ (\sim_k\rightarrow_j\text{left}).$$

■

LEMMA 5.10

Let f be the mapping defined in Definition 5.6 and let $\mathcal{A} := \langle U, I^p\rangle$ be a paraconsistent model. For any paraconsistent satisfaction relation $\models^p$ on $\mathcal{A}$, we can construct a satisfaction relation $\models$ on a model $\mathcal{A}' = \langle U, I\rangle$ such that for any formula α,

$\mathcal{A} \models^p \alpha$ iff $\mathcal{A}' \models f(\alpha)$.

PROOF. Let Φ be a set of atomic formulas and for every $j \leq n$ let Φ^j be the set $\{p^j \mid p \in \Phi\}$ of atomic formulas. Suppose that $\models^p$ is a paraconsistent satisfaction relation on $\mathcal{A}$. Suppose that $\models$ is a satisfaction relation on $\mathcal{A}'$ such that, for any atomic formula $p \in \Phi \cup \bigcup_j \Phi^j$,

(1) $\mathcal{A} \models^p p$ iff $\mathcal{A}' \models p$,
(2) $\mathcal{A} \models^p \sim_j p$ iff $\mathcal{A}' \models p^j$.

Then, the lemma is proved by induction on α.

- Base step:

(1) Case when α is p, where p is a propositional variable: $\mathcal{A} \models^p p$ iff $\mathcal{A}' \models p$ (by the assumption) iff $\mathcal{A}' \models f(p)$ (by the definition of f).
(2) Case when α is $\sim_j p$, where p is a propositional variable: $\mathcal{A} \models^p \sim_j p$ iff $\mathcal{A}' \models p^j$ (by the assumption) iff $\mathcal{A}' \models f(\sim_j p)$ (by the definition of f).

- Induction step: We show some cases.

(1) Case when α is $\beta \rightarrow_j \gamma$: $\mathcal{A} \models^p \beta \rightarrow_j \gamma$ iff not-($\mathcal{A} \models^p \beta$) or $\mathcal{A} \models^p \gamma$ iff not-($\mathcal{A}' \models f(\beta)$) or $\mathcal{A}' \models f(\gamma)$ (by induction hypothesis) iff $\mathcal{A}' \models f(\beta) \rightarrow f(\gamma)$ iff $\mathcal{A}' \models f(\beta \rightarrow_j \gamma)$ (by the definition of f).
(2) Case when α is $\beta \leftarrow_j \gamma$: $\mathcal{A} \models^p \beta \leftarrow_j \gamma$ iff $\mathcal{A} \models^p \beta$ and $\mathcal{A} \not\models^p \gamma$ iff $\mathcal{A}' \models f(\beta)$ and $\mathcal{A}' \not\models f(\gamma)$ (by induction hypothesis) iff $\mathcal{A}' \models f(\beta) \leftarrow f(\gamma)$ iff $\mathcal{A}' \models f(\beta \leftarrow_j \gamma)$ (by the definition of f).

(3) Case when α is $\forall_j x\beta$: $\mathcal{A} \models^p \forall_j x\beta$ iff $\mathcal{A} \models^p \beta[\underline{u}/x]$ for all $u \in U$ iff $\mathcal{A}' \models f(\beta[\underline{u}/x])$ for all $u \in U$ iff (by induction hypothesis) iff $\mathcal{A}' \models \forall x f(\beta)$ iff $\mathcal{A}' \models f(\forall_j x\beta)$ (by the definition of f).

(4) Case when α is $\sim_j\sim_j\beta$: $\mathcal{A} \models^p \sim_j\sim_j\beta$ iff $\mathcal{A} \models^p \beta$ iff $\mathcal{A}' \models f(\beta)$ iff (by induction hypothesis) iff $\mathcal{A}' \models f(\sim_j\sim_j\beta)$ (by the definition of f).

(5) Case when α is $\sim_j(\beta \rightarrow_j \gamma)$: $\mathcal{A} \models^p \sim_j(\beta \rightarrow_j \gamma)$ iff $\mathcal{A} \models^p \sim_j\gamma$ and $\mathcal{A} \not\models^p \sim_j\beta$ iff $\mathcal{A}' \models f(\sim_j\gamma)$ and $\mathcal{A}' \not\models f(\sim_j\beta)$ (by induction hypothesis) iff $\mathcal{A}' \models f(\sim_j\gamma) \leftarrow f(\sim_j\beta)$ iff $\mathcal{A}' \models f(\sim_j(\beta \rightarrow_j \gamma))$ (by the definition of f).

(6) Case when α is $\sim_j(\beta \leftarrow_j \gamma)$: $\mathcal{A} \models^p \sim_j(\beta \leftarrow_j \gamma)$ iff $\mathcal{A} \not\models^p \sim_j\gamma$ or $\mathcal{A} \models^p \sim_j\beta$ iff $\mathcal{A}' \not\models f(\sim_j\gamma)$ or $\mathcal{A}' \models f(\sim_j\beta)$ (by induction hypothesis) iff $\mathcal{A}' \models f(\sim_j\gamma) \rightarrow f(\sim_j\beta)$ iff $\mathcal{A}' \models f(\sim_j(\beta \leftarrow_j \gamma))$ (by the definition of f).

(7) Case when α is $\sim_j\forall_j x\beta$: $\mathcal{A} \models^p \sim_j\forall_j x\beta$ iff $\mathcal{A} \models^p \sim_j\beta[\underline{u}/x]$ for some $u \in U$ iff $\mathcal{A}' \models f(\sim_j\beta[\underline{u}/x])$ for some $u \in U$ iff (by induction hypothesis) iff $\mathcal{A}' \models \exists x f(\sim_j\beta)$ iff $\mathcal{A}' \models f(\sim_j\forall_j x\beta)$ (by the definition of f).

(8) Case when α is $\sim_k\sim_j\beta$: $\mathcal{A} \models^p \sim_k\sim_j\beta$ iff $\mathcal{A} \not\models^p \beta$ iff $\mathcal{A}' \not\models f(\beta)$ (by induction hypothesis) iff $\mathcal{A}' \models \neg f(\beta)$ iff $\mathcal{A}' \models f(\sim_k\sim_j\beta)$ (by the definition of f).

(9) Case when α is $\sim_k(\beta \rightarrow_j \gamma)$: $\mathcal{A} \models^p \sim_k(\beta \rightarrow_j \gamma)$ iff $\mathcal{A} \not\models^p \sim_k\beta$ or $\mathcal{A} \models^p \sim_k\gamma$ iff $\mathcal{A}' \not\models f(\sim_k\beta)$ or $\mathcal{A}' \models f(\sim_k\gamma)$ (by induction hypothesis) iff $\mathcal{A}' \models f(\sim_k\beta) \rightarrow f(\sim_k\gamma)$ iff $\mathcal{A}' \models f(\sim_k(\beta \rightarrow_j \gamma)$ (by the definition of f).

(10) Case when α is $\sim_k(\beta \leftarrow_j \gamma)$: $\mathcal{A} \models^p \sim_k(\beta \leftarrow_j \gamma)$ iff $\mathcal{A} \models^p \sim_k\beta$ and $\mathcal{A} \not\models^p \sim_k\gamma$ iff $\mathcal{A}' \models f(\sim_k\beta)$ and $\mathcal{A}' \not\models f(\sim_k\gamma)$ (by induction hypothesis) iff $\mathcal{A}' \models f(\sim_k\beta) \leftarrow f(\sim_k\gamma)$ iff $\mathcal{A}' \models f(\sim_k(\beta \leftarrow_j \gamma))$ (by the definition of f).

(11) Case when α is $\sim_k\forall_j x\beta$: $\mathcal{A} \models^p \sim_k\forall_j x\beta$ iff $\mathcal{A} \models^p \sim_k\beta[\underline{u}/x]$ for all $u \in U$ iff $\mathcal{A}' \models f(\sim_k\beta[\underline{u}/x])$ for all $u \in U$ iff (by induction hypothesis) iff $\mathcal{A}' \models \forall x f(\sim_k\beta)$ iff $\mathcal{A}' \models f(\sim_k\forall_j x\beta)$ (by the definition of f).

■

6 Summarizing remarks

The embedding-based proof methods used in this article can be adapted to a wide range of non-classical logics. In this respect, it is instructive to refer readers to some closely related works on the embedding-based approaches in non-classical logics and briefly compare them with the methods developed in the present article.

The syntactical embedding of the classical paraconsistent logic L_ω into classical logic was studied in [9], where its applicability for obtaining the cut-elimination theorem for L_ω was shown. The proof techniques of the embedding and cut-elimination theorems for L_ω are almost the same as those of ML_n. However, [9] employs a standard completeness proof for L_ω, as distinct from the method developed above, where we use the embedding-based proof technique to establish the completeness theorem for ML_n (and FML_n).

The syntactical and semantical embeddings of *Nelson's paraconsistent four-valued logic* N4 and its neighbours into their positive fragments were studied and in [16]. THis survey paper explained that the completeness theorems with respect to Kripke semantics for N4 and its neighbours could be obtained by using both the syntactical and semantical embedding theorems. It should also be mentioned that the embedding-based completeness proof for N4 was originally obtained in [10].

The Craig interpolation theorem for a variant of N4 was studied in [11], where it was shown that the syntactical embedding of this variant into its paraconsistent negation-free fragment could be used to prove the theorem. Additionally, the syntactical and semantical embedding theorems of some many-valued paraconsistent logics into their positive fragments were studied in [13], which showed that the embedding theorems could be used to investigate a hierarchy of weak double negation axioms.

In this article, we have mainly employed the embedding-based proof technique used in [14] (which is an extended version of the conference paper [12]), where the syntactical and semantical embeddings of some *trilattice logics* into the classical logic, and their usefulness for obtaining the cut-elimination, completeness and Craig interpolation theorems for these trilattice logics were carefully studied (for the notion of a trilattice readers are referred to [20]). For example, Theorems 2.4 and 2.6 here correspond to Theorems 3.4 and 3.6 in [14], whereas Lemmas 3.3, 3.4 and Theorem 3.5 correspond to Lemmas 4.3, 4.4 and Theorem 4.5.

The embedding-based methods have been further developed to reveal their full generality and applicability in establishing some important metaproperties of the generalized multilattice logics. Most crucially, the embedding theorems in the present article are essentially 'bi-directional'. Namely, they describe translations not only from ML_n into LK, but also from LK into ML_n. This is in contrast to the approach developed in [14], where only the embeddings from the logics under consideration into their negation-free fragments were introduced.

In particular, the translation functions defined from LK into ML_n, and vice versa allow us to represent the classical negation $\neg$ as a composition of two paraconsistent negations $\sim_k\sim_j$, thus interpreting classical negation as a kind of 'paraconsistent double negation'. In this way, the bi-directional interpretations establish an interesting relationship between the paraconsistent and classical negations, presenting a complete characterization of the paraconsistent double negations by the classical one, and vice versa. These bi-directional interpretations are also useful for formalizing and proving the modified Craig interpolation theorem for ML_n, which is difficult by other means. Moreover, Definition 2.8 shows that there are in fact various distinct translations of LK into ML_n depending on the choice of j and k in this definition.

Finally, Section 5 shows how the results for ML_n can be straightforwardly extended to its first-order version FML_n with implications and co-implications, which is another novel contribution of the present study.

7 Conclusion

The multilattice logic presents an appropriate framework for simultaneously handling several (in fact n) distinct, although uniform, connectives of a certain kind. In [24, p. 496], the case with the duplication of logical connectives is considered by splitting each connective into its 'positive' and 'negative' counterparts, depending on whether they are defined in terms of truth or falsity, where the later notions are no longer complementary. If, however, by the specification of connectives we involve some additional characteristics, such as information, modality or constructivity, then a natural multiplication of entities seems not only to be possible, but even necessary, thus fulfilling the requirements of Occam's razor.

Although the main motivation for the multilattice logic initially comes from specific algebraic structures determined on the sets of truth values by various ordering relations, it is most telling that the central metaproperties of the resulting logical systems, such as soundness, completeness, decidability and cut-elimination, can be established by 'pure logical' means with no recourse to algebraic machinery. This fact indicates the logical importance of the multilattice systems themselves as a general deductive construction that is both designed and suitable for an analysis of reasoning patterns that may require various subtle distinctions in the meanings of the connectives.

Moreover, the two-valued semantics for the multilattice logic formulated by Definition 3.1 may serve as another illustration of the famous Suszko Thesis with its crucial distinction between logical and algebraic values, see, e.g. [25], at least as a particular case of its realization, even though the thesis itself (in its full generality) remains hotly disputed in modern logical debates.

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