Log. Univers.

DOI 10.1007/s11787-017-0172-5

Logica Universalis

Modal Multilattice Logic

Norihiro Kamide and Yaroslav Shramko

Abstract. A modal extension of multilattice logic, called modal multilattice logic, is introduced as a Gentzen-type sequent calculus MML_n. Theorems for embedding MML_n into a Gentzen-type sequent calculus S4C (an extended S4-modal logic) and vice versa are proved. The cut-elimination theorem for MML_n is shown. A Kripke semantics for MML_n is introduced, and the completeness theorem with respect to this semantics is proved. Moreover, the duality principle is proved as a characteristic property of MML_n.

Mathematics Subject Classification. Primary 03B45; Secondary 03B53.

Keywords. Multilattice logic, modal logic, embedding theorem, completeness theorem, sequent calculus, duality.

1. Introduction

The aim of this paper is to extend to the realm of modal logics the notion of a multilattice logic, introduced in [30] and studied further in [17]. The idea of a multilattice logics, stems essentially from *Belnap and Dunn's useful four-valued logic* [3,4,7], and is also a generalization of *Arieli and Avron's bilattice logics* [2], *Shramko and Wansing's trilattice logics* [33,34], and *Zaitsev's tetralattice logic* [38]. Semantically these logics are based on certain algebraic structures, such as *bilattices* [10,11], *trilattices* [32,33], *tetralattices* [38] and beyond, generalized in [30] under a joint name of *multilattices*.

Generally, an n-dimensional multilattice (or just n-lattice) is a lattice equipped with exactly n partial orders. It was argued in [30] that multilattices present a natural algebraic framework for *generalized truth-values*, conceived as elements of the power-set of some (basic) set of initial truth-values. Every partial order in a given multilattice serves as a tool for gradating the respective

Norihiro Kamide was supported by JSPS KAKENHI Grant (C) JP26330263. Yaroslav Shramko's work on this paper was a part of Marie Curie project PIRSES-GA-2012-318986 *Generalizing Truth Functionality* within the Seventh Framework Programme for Research funded by EU.

Published online: 20 July 2017

Birkhäuser

generalized truth values, in accordance with a certain characterization, such as information, truth, falsity, constructivity, (un)certainty, modality, or other kinds of "adverbial qualifications", see [30, p. 205]. Meet and join exist for any such order, which can also be supplied with an appropriate inversion operator. Thus, an n-lattice can be seen as a basis for the corresponding multilattice logics with exactly n pairs of conjunctions and disjunctions, and also with n negation-like operators.

In this paper, a new logic called *modal multilattice logic* is introduced as a Gentzen-type sequent calculus MML_n. This logic is an extension (by n pairs of modal operators for every partial ordering) of the multilattice logic (with n pairs of conjunctions, disjunctions and n paraconsistent negations) originally introduced in [30], and deductively formalized there by a cut-free Gentzen-type sequent calculus GML_n. In [17] it was slightly modified by a Gentzen-type sequent calculus ML_n. Moreover, an embedding technique (for mutual translating between formulas of multilattice logic and classical logic) was effectively applied there for establishing some key results, such as the cut-elimination, decidability and completeness of ML_n, as well as modified Craig interpolation and Maksimova separation theorems. Besides, a first-order extension of ML_n was formulated by adding implications, co-implications, universal quantifiers and existential quantifiers. It was shown in [17] that the same theorems as those for ML_n can be proved for this extension.

The main contribution of the present study is a construction of a simple and natural Gentzen-type sequent calculus MML_n for the propositional multilattice logic with S4-like modalities. By using an embedding technique this calculus is shown to be decidable, complete and cut-free. Moreover, it is proved that the duality principle as a characteristic property of the multilattice logics holds for MML_n and its fragments.

The structure of this paper can be summarized as follows. In Sect. 2 we remind some key definitions connected to the notion of logical multilattice and motivate its modal extension. In Sect. 3, two sequent calculi MML_n and S4C are introduced, and two translation functions from MML_n into S4C and vice versa are defined. In Sect. 4, several theorems for a syntactic embedding MML_n into S4C and vice versa are proved, and the cut-elimination theorem for MML_n is shown. MML_n is also shown to be decidable, and it is remarked that modified Craig interpolation theorem and Maksimova separation theorem hold. In Sect. 5, a Kripke semantics for MML_n is introduced, and several theorems for a semantic embedding MML_n into S4C and vice versa are proved, establishing thus the completeness of MML_n with respect to this semantics. In Sect. 6, the duality principle for MML_n is proved.

2. Logical Multilattices and their Modal Extensions

Let us first recall the definition of a multilattice from [30] as an algebraic structure with several partial orderings:

Definition 2.1. An n-dimensional *multilattice* (or just n-lattice) is a structure

$$\mathcal{M}_n = (S, \sqsubseteq_1, \ldots, \sqsubseteq_n),$$

where S is a non-empty set, and $\sqsubseteq_1, \ldots, \sqsubseteq_n$ are partial orderings each giving S the structure of a lattice, determining thus for each of the n lattices the corresponding pairs of meet and join operations denoted by $\langle \sqcap_1, \sqcup_1 \rangle, \ldots, \langle \sqcap_n, \sqcup_n \rangle$.

Besides meets and joins, a multilattice can be equipped with inversion operations defined with respect to each ordering relation:

Definition 2.2. Let $\mathcal{M}_n = (S, \sqsubseteq_1, \ldots, \sqsubseteq_n)$ be a multilattice. Then for any $j \leq n$ an unary operation $-_j$ on S is said to be a (pure) *j-inversion* iff for any $k \leq n$, $k \neq j$ the following conditions are satisfied:

$$\begin{array}{ll} (anti) & x \sqsubseteq_j y \Rightarrow -_j y \sqsubseteq_j -_j x; \\ (iso) & x \sqsubseteq_k y \Rightarrow -_j x \sqsubseteq_k -_j y; \\ (per2) & -_j -_j x = x. \end{array}$$

The following notion of a multifilter is a generalization of the notion of a bifilter from [2]:

Definition 2.3. Let $\mathcal{M}_n = (S, \sqsubseteq_1, \ldots, \sqsubseteq_n)$ be an n-lattice, with pairs of meet and join operations $\langle \sqcap_1, \sqcup_1 \rangle, \ldots, \langle \sqcap_n, \sqcup_n \rangle$. An *n-filter* (multifilter) on $\mathcal{M}_n$ is a nonempty proper subset $\mathcal{F}_n \subset S$, such that for every $j \leq n$:

$$x \sqcap_j y \in \mathcal{F}_n \Leftrightarrow x \in \mathcal{F}_n \text{ and } y \in \mathcal{F}_n.$$

A multifilter $\mathcal{F}_n$ is said to be *prime* iff it satisfies for every $j \leq n$:

$$x \sqcup_j y \in \mathcal{F}_n \Leftrightarrow x \in \mathcal{F}_n \text{ or } y \in \mathcal{F}_n.$$

A pair $(\mathcal{M}_n, \mathcal{F}_n)$ is called a *logical n-lattice* (logical multilattice) iff $\mathcal{M}_n$ is a multilattice, and $\mathcal{F}_n$ is a prime multifilter on $\mathcal{M}_n$.

If we are interested in n-lattices with inversions existing for every $j \leq n$, we can consider the stronger notions of an ultramultifilter and ultralogical multilattice.

Definition 2.4. Let $\mathcal{M}_n = (S, \sqsubseteq_1, \ldots, \sqsubseteq_n)$ be an n-lattice, with j-inversions defined with respect to every $\sqsubseteq_j$ ($j \leq n$). Then $\mathcal{F}_n$ is an *n-ultrafilter* (ultramultifilter) on $\mathcal{M}_n$ if and only if it is a prime multifilter, such that for every $j, k \leq n$, $j \neq k$: $x \in \mathcal{F}_n \Leftrightarrow -_j -_k x \notin \mathcal{F}_n$. A pair $(\mathcal{M}_n, \mathcal{F}_n)$ is called an *ultralogical n-lattice* (ultralogical multilattice) iff $\mathcal{M}_n$ is a multilattice, and $\mathcal{F}_n$ is an ultramultifilter on $\mathcal{M}_n$.

Intuitively meet, join and inversion with respect to an ordering in a given multilattice determine the corresponding connectives of conjunction, disjunction and negation. Thus, an n-lattice with inversions generates exactly n basic pairs of conjunctions and disjunctions, accompanied with at least n negation-like operators. The respective (ultra)multifilter represents then the set of designated elements used for defining the corresponding entailment relation. Having an ultralogical multilattice, the corresponding *minimal multilattice logic* can be conceived as a system which operates solely with the connectives of conjunctions, disjunctions and negations. For any n (>1), such multilattice logic was formalized in [30] as a Gentzen-style sequent system GML_n. Some features of this system were investigated further in [17], where its slightly modified version ML_n was presented.

Moreover, in [17] this minimal multilattice logic was extended by quantifiers, implications and co-implications, determined for each $j \leq n$ by their classical-type inference rules. In [19] system ML_n was extended by the connectives of intuitionistic and dual-intuitionistic implications, resulting thus in the bi-intuitionistic multilattice logic. Moreover, another logic called *bi-intuitionistic connexive multilattice logic*, was obtained there by replacing the connectives of intuitionistic implication and co-implication with their connexive variants.

In the next section we show how system ML_n can be suitably extended to obtain the modal multilattice logic MML_n. Namely, for every partial order $\leq_j$ in a given multilattice we consider the object language operators for necessity $\Box_j$ and possibility $\Diamond_j$ of S4-type. Algebraically this means an equipment of the corresponding multilattices with Tarski interior and closure operations (cf. [21,22], see also [6]). We can thus consider the notion of a *modal multilattice* defined as follows:

Definition 2.5. A multilattice $\mathcal{M}_n = (S, \sqsubseteq_1, \ldots, \sqsubseteq_n)$ is said to be *modal* iff for any $j \leq n$ the unary operations of *interior* $\mathbf{I}_j$ and *closure* $\mathbf{C}_j$ can be defined on S, satisfying the following conditions:

$$
\begin{array}{ll}
(\textit{decreasing}) & \mathbf{I}_j(x) \sqsubseteq_j x; \\
(\textit{idempotent}) & \mathbf{I}_j(x) = \mathbf{I}_j\mathbf{I}_j(x); \\
(\textit{sub-multiplicative}) & \mathbf{I}_j(x \sqcap_j y) \sqsubseteq_j \mathbf{I}_j(x) \sqcap_j \mathbf{I}_j(y).
\end{array}
$$

$$
\begin{array}{ll}
(\textit{increasing}) & x \sqsubseteq_j \mathbf{C}_j(x); \\
(\textit{idempotent}) & \mathbf{C}_j(x) = \mathbf{C}_j\mathbf{C}_j(x); \\
(\textit{sub-additive}) & \mathbf{C}_j(x) \sqcup_j \mathbf{C}_j(y) \sqsubseteq_j \mathbf{C}_j(x \sqcup_j y).
\end{array}
$$

It is well known that operations of interior and closure are algebraic counterparts of the necessity and possibility operators of S4-type, see, e.g., [6, p. 97]. Moreover, Lemmon [20] observes close connection between modal algebras and Kripke model structures. In particular, he shows how one can obtain natural representation theorems for his modal algebras in terms of different model structures, and consequently gets the Kripke-type completeness results for various modal logics [20, p. 56].

In what follows, we pursue a model-theoretic approach to the modal multilattice logic, and establish decidability, completeness and some other important results by using an embedding technique. The exact coupling the paraconsistent Kripke models formulated in this paper with the underlying algebraic structures of multilattices is a matter for future work.

3. Sequent Calculi and Translation Functions

Let n (>1) be the positive integer determined by an n-lattice. Then, *formulas* of modal multilattice (n-lattice) logic are defined using countably many propositional variables, logical connectives $\wedge_j$, $\vee_j$, $\rightarrow_j$, $\leftarrow_j$, $\sim_j$, and modal operators $\Box_j$, $\Diamond_j$, for every $j \leq n$. We use small letters $p, q, ...$ to denote propositional variables, Greek small letters $\alpha, \beta, ...$ to denote formulas, and Greek capital letters $\Gamma, \Delta, ...$ to denote finite (possibly empty) sets of formulas. We use an expression $\sharp\Gamma$ ($\sharp \in \{\sim_j, \Box_j, \Diamond_j\}$) to denote the set $\{\sharp\gamma \mid \gamma \in \Gamma\}$. A *sequent* is an expression of the form $\Gamma \Rightarrow \Delta$. We use an expression $\alpha \Leftrightarrow \beta$ to represent the sequents $\alpha \Rightarrow \beta$ and $\beta \Rightarrow \alpha$, and an expression $L \vdash \Gamma \Rightarrow \Delta$ to represent the fact that the sequent $\Gamma \Rightarrow \Delta$ is provable in a Gentzen-type sequent calculus L.

A Gentzen-type sequent calculus MML_n for the modal multilattice logic is defined as follows.

Definition 3.1 (*MML_n*). Let n (>1) be the positive integer determined by n-lattice, and j, k be any positive integers with $j, k \leq n$ and $j \neq k$.

The initial sequents of MML_n are of the following form, for any propositional variable p,

$$p \Rightarrow p \qquad\qquad \sim_j p \Rightarrow \sim_j p.$$

The structural inference rules of MML_n are of the form:

$$\frac{\Gamma \Rightarrow \Delta, \alpha \quad \alpha, \Sigma \Rightarrow \Pi}{\Gamma, \Sigma \Rightarrow \Delta, \Pi}\ (\text{cut}) \qquad \frac{\Gamma \Rightarrow \Delta}{\alpha, \Gamma \Rightarrow \Delta}\ (\text{we-left}) \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \alpha}\ (\text{we-right}).$$

The non-negated logical inference rules of MML_n are of the form:

$$\frac{\alpha, \beta, \Gamma \Rightarrow \Delta}{\alpha\wedge_j\beta, \Gamma \Rightarrow \Delta}\ (\wedge_j\text{left}) \qquad \frac{\Gamma \Rightarrow \Delta, \alpha \quad \Gamma \Rightarrow \Delta, \beta}{\Gamma \Rightarrow \Delta, \alpha\wedge_j\beta}\ (\wedge_j\text{right})$$

$$\frac{\alpha, \Gamma \Rightarrow \Delta \quad \beta, \Gamma \Rightarrow \Delta}{\alpha\vee_j\beta, \Gamma \Rightarrow \Delta}\ (\vee_j\text{left}) \qquad \frac{\Gamma \Rightarrow \Delta, \alpha, \beta}{\Gamma \Rightarrow \Delta, \alpha\vee_j\beta}\ (\vee_j\text{right})$$

$$\frac{\Gamma \Rightarrow \Delta, \alpha \quad \beta, \Sigma \Rightarrow \Pi}{\alpha\rightarrow_j\beta, \Gamma, \Sigma \Rightarrow \Delta, \Pi}\ (\rightarrow_j\text{left}) \qquad \frac{\alpha, \Gamma \Rightarrow \Delta, \beta}{\Gamma \Rightarrow \Delta, \alpha\rightarrow_j\beta}\ (\rightarrow_j\text{right})$$

$$\frac{\alpha, \Gamma \Rightarrow \Delta, \beta}{\alpha\leftarrow_j\beta, \Gamma \Rightarrow \Delta}\ (\leftarrow_j\text{left}) \qquad \frac{\Gamma \Rightarrow \Delta, \alpha \quad \beta, \Sigma \Rightarrow \Pi}{\Gamma, \Sigma \Rightarrow \Delta, \Pi, \alpha\leftarrow_j\beta}\ (\leftarrow_j\text{right}).$$

The jj-negated logical inference rules of MML_n are of the form:

$$\frac{\sim_j\alpha, \Gamma \Rightarrow \Delta \quad \sim_j\beta, \Gamma \Rightarrow \Delta}{\sim_j(\alpha\wedge_j\beta), \Gamma \Rightarrow \Delta}\ (\sim_j\wedge_j\text{left}) \qquad \frac{\Gamma \Rightarrow \Delta, \sim_j\alpha, \sim_j\beta}{\Gamma \Rightarrow \Delta, \sim_j(\alpha\wedge_j\beta)}\ (\sim_j\wedge_j\text{right})$$

$$\frac{\sim_j\alpha, \sim_j\beta, \Gamma \Rightarrow \Delta}{\sim_j(\alpha\vee_j\beta), \Gamma \Rightarrow \Delta}\ (\sim_j\vee_j\text{left}) \qquad \frac{\Gamma \Rightarrow \Delta, \sim_j\alpha \quad \Gamma \Rightarrow \Delta, \sim_j\beta}{\Gamma \Rightarrow \Delta, \sim_j(\alpha\vee_j\beta)}\ (\sim_j\vee_j\text{right})$$

$$\frac{\sim_j\beta, \Gamma \Rightarrow \Delta, \sim_j\alpha}{\sim_j(\alpha\rightarrow_j\beta), \Gamma \Rightarrow \Delta}\ (\sim_j\rightarrow_j\text{left}) \qquad \frac{\Gamma \Rightarrow \Delta, \sim_j\beta \quad \sim_j\alpha, \Sigma \Rightarrow \Pi}{\Gamma, \Sigma \Rightarrow \Delta, \Pi, \sim_j(\alpha\rightarrow_j\beta)}\ (\sim_j\rightarrow_j\text{right})$$

$$\frac{\Gamma \Rightarrow \Delta, \sim_j\beta \quad \sim_j\alpha, \Sigma \Rightarrow \Pi}{\sim_j(\alpha\leftarrow_j\beta), \Gamma, \Sigma \Rightarrow \Delta, \Pi}\ (\sim_j\leftarrow_j\text{left}) \qquad \frac{\sim_j\beta, \Gamma \Rightarrow \Delta, \sim_j\alpha}{\Gamma \Rightarrow \Delta, \sim_j(\alpha\leftarrow_j\beta)}\ (\sim_j\leftarrow_j\text{right})$$

$$\frac{\alpha, \Gamma \Rightarrow \Delta}{\sim_j\sim_j\alpha, \Gamma \Rightarrow \Delta}\ (\sim_j\sim_j\text{left}) \qquad \frac{\Gamma \Rightarrow \Delta, \alpha}{\Gamma \Rightarrow \Delta, \sim_j\sim_j\alpha}\ (\sim_j\sim_j\text{right}).$$

The kj-negated logical inference rules of MML_n are of the form:

$$\frac{\sim_k\alpha, \sim_k\beta, \Gamma \Rightarrow \Delta}{\sim_k(\alpha\wedge_j\beta), \Gamma \Rightarrow \Delta}\ (\sim_k\wedge_j\text{left}) \qquad \frac{\Gamma \Rightarrow \Delta, \sim_k\alpha \quad \Gamma \Rightarrow \Delta, \sim_k\beta}{\Gamma \Rightarrow \Delta, \sim_k(\alpha\wedge_j\beta)}\ (\sim_k\wedge_j\text{right})$$

$$\frac{\sim_k\alpha, \Gamma \Rightarrow \Delta \quad \sim_k\beta, \Gamma \Rightarrow \Delta}{\sim_k(\alpha\vee_j\beta), \Gamma \Rightarrow \Delta}\ (\sim_k\vee_j\text{left}) \qquad \frac{\Gamma \Rightarrow \Delta, \sim_k\alpha, \sim_k\beta}{\Gamma \Rightarrow \Delta, \sim_k(\alpha\vee_j\beta)}\ (\sim_k\vee_j\text{right})$$

$$\frac{\Gamma \Rightarrow \Delta, \sim_k\alpha \quad \sim_k\beta, \Sigma \Rightarrow \Pi}{\sim_k(\alpha\rightarrow_j\beta), \Gamma, \Sigma \Rightarrow \Delta, \Pi}\ (\sim_k\rightarrow_j\text{left}) \qquad \frac{\sim_k\alpha, \Gamma \Rightarrow \Delta, \sim_k\beta}{\Gamma \Rightarrow \Delta, \sim_k(\alpha\rightarrow_j\beta)}\ (\sim_k\rightarrow_j\text{right})$$

$$\frac{\sim_k\alpha, \Gamma \Rightarrow \Delta, \sim_k\beta}{\sim_k(\alpha\leftarrow_j\beta), \Gamma \Rightarrow \Delta}\ (\sim_k\leftarrow_j\text{left}) \qquad \frac{\Gamma \Rightarrow \Delta, \sim_k\alpha \quad \sim_k\beta, \Sigma \Rightarrow \Pi}{\Gamma, \Sigma \Rightarrow \Delta, \Pi, \sim_k(\alpha\leftarrow_j\beta)}\ (\sim_k\leftarrow_j\text{right})$$

$$\frac{\Gamma \Rightarrow \Delta, \alpha}{\sim_k\sim_j\alpha, \Gamma \Rightarrow \Delta}\ (\sim_k\sim_j\text{left}) \qquad \frac{\alpha, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \sim_k\sim_j\alpha}\ (\sim_k\sim_j\text{right}).$$

The non-negated modal inference rules of MML_n are of the form:

$$\frac{\alpha, \Gamma \Rightarrow \Delta}{\Box_j\alpha, \Gamma \Rightarrow \Delta}\ (\Box_j\text{left}) \qquad \frac{\Box_j\Gamma, \sim_j\Diamond_j\Sigma, \sim_k\Box_j\Pi \Rightarrow \alpha}{\Box_j\Gamma, \sim_j\Diamond_j\Sigma, \sim_k\Box_j\Pi \Rightarrow \Box_j\alpha}\ (\Box_j\text{right})$$

$$\frac{\alpha \Rightarrow \Diamond_j\Gamma, \sim_j\Box_j\Sigma, \sim_k\Diamond_j\Pi}{\Diamond_j\alpha \Rightarrow \Diamond_j\Gamma, \sim_j\Box_j\Sigma, \sim_k\Diamond_j\Pi}\ (\Diamond_j\text{left}) \qquad \frac{\Gamma \Rightarrow \Delta, \alpha}{\Gamma \Rightarrow \Delta, \Diamond_j\alpha}\ (\Diamond_j\text{right}).$$

The jj-negated modal inference rules of MML_n are of the form:

$$\frac{\sim_j\alpha \Rightarrow \Diamond_j\Gamma, \sim_j\Box_j\Sigma, \sim_k\Diamond_j\Pi}{\sim_j\Box_j\alpha \Rightarrow \Diamond_j\Gamma, \sim_j\Box_j\Sigma, \sim_k\Diamond_j\Pi}\ (\sim_j\Box_j\text{left}) \qquad \frac{\Gamma \Rightarrow \Delta, \sim_j\alpha}{\Gamma \Rightarrow \Delta, \sim_j\Box_j\alpha}\ (\sim_j\Box_j\text{right})$$

$$\frac{\sim_j\alpha, \Gamma \Rightarrow \Delta}{\sim_j\Diamond_j\alpha, \Gamma \Rightarrow \Delta}\ (\sim_j\Diamond_j\text{left}) \qquad \frac{\Box_j\Gamma, \sim_j\Diamond_j\Sigma, \sim_k\Box_j\Pi \Rightarrow \sim_j\alpha}{\Box_j\Gamma, \sim_j\Diamond_j\Sigma, \sim_k\Box_j\Pi \Rightarrow \sim_j\Diamond_j\alpha}\ (\sim_j\Diamond_j\text{right}).$$

The kj-negated modal inference rules of MML_n are of the form:

$$\frac{\sim_k\alpha, \Gamma \Rightarrow \Delta}{\sim_k\Box_j\alpha, \Gamma \Rightarrow \Delta}\ (\sim_k\Box_j\text{left}) \qquad \frac{\Box_j\Gamma, \sim_j\Diamond_j\Sigma, \sim_k\Box_j\Pi \Rightarrow \sim_k\alpha}{\Box_j\Gamma, \sim_j\Diamond_j\Sigma, \sim_k\Box_j\Pi \Rightarrow \sim_k\Box_j\alpha}\ (\sim_k\Box_j\text{right})$$

$$\frac{\sim_k\alpha \Rightarrow \Diamond_j\Gamma, \sim_j\Box_j\Sigma, \sim_k\Diamond_j\Pi}{\sim_k\Diamond_j\alpha \Rightarrow \Diamond_j\Gamma, \sim_j\Box_j\Sigma, \sim_k\Diamond_j\Pi}\ (\sim_k\Diamond_j\text{left}) \qquad \frac{\Gamma \Rightarrow \Delta, \sim_k\alpha}{\Gamma \Rightarrow \Delta, \sim_k\Diamond_j\alpha}\ (\sim_k\Diamond_j\text{right}).$$

A Gentzen-type sequent calculus S4C for the extended S4-normal modal logic with co-implication is introduced below. Formulas of S4C are constructed from countably many propositional variables, logical connectives $\wedge$, $\vee$, $\rightarrow$, $\leftarrow$, $\neg$, and modal operators $\Box$, $\Diamond$.

Definition 3.2 (*S4C*). The initial sequents of S4C are of the following form, for any propositional variable p,

$$p \Rightarrow p.$$

The structural inference rules of S4C are the same as those of MML_n.

The logical inference rules of S4C are of the form:

$$\frac{\alpha, \beta, \Gamma \Rightarrow \Delta}{\alpha \wedge \beta, \Gamma \Rightarrow \Delta}\ (\wedge\text{left}) \qquad \frac{\Gamma \Rightarrow \Delta, \alpha \quad \Gamma \Rightarrow \Delta, \beta}{\Gamma \Rightarrow \Delta, \alpha \wedge \beta}\ (\wedge\text{right})$$

$$\frac{\alpha, \Gamma \Rightarrow \Delta \quad \beta, \Gamma \Rightarrow \Delta}{\alpha \vee \beta, \Gamma \Rightarrow \Delta}\ (\vee\text{left}) \qquad \frac{\Gamma \Rightarrow \Delta, \alpha, \beta}{\Gamma \Rightarrow \Delta, \alpha \vee \beta}\ (\vee\text{right})$$

$$\frac{\Gamma \Rightarrow \Delta, \alpha \quad \beta, \Sigma \Rightarrow \Pi}{\alpha{\rightarrow}\beta, \Gamma, \Sigma \Rightarrow \Delta, \Pi}\ ({\rightarrow}\text{left}) \qquad \frac{\alpha, \Gamma \Rightarrow \Delta, \beta}{\Gamma \Rightarrow \Delta, \alpha{\rightarrow}\beta}\ ({\rightarrow}\text{right})$$

$$\frac{\alpha, \Gamma \Rightarrow \Delta, \beta}{\alpha{\leftarrow}\beta, \Gamma \Rightarrow \Delta}\ ({\leftarrow}\text{left}) \qquad \frac{\Gamma \Rightarrow \Delta, \alpha \quad \beta, \Sigma \Rightarrow \Pi}{\Gamma, \Sigma \Rightarrow \Delta, \Pi, \alpha{\leftarrow}\beta}\ ({\leftarrow}\text{right})$$

$$\frac{\Gamma \Rightarrow \Delta, \alpha}{\neg\alpha, \Gamma \Rightarrow \Delta}\ (\neg\text{left}) \qquad \frac{\alpha, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg\alpha}\ (\neg\text{right}).$$

The modal inference rules of S4C are of the form:

$$\frac{\alpha, \Gamma \Rightarrow \Delta}{\Box\alpha, \Gamma \Rightarrow \Delta}\ (\Box\text{left}) \qquad \frac{\Box\Gamma \Rightarrow \alpha}{\Box\Gamma \Rightarrow \Box\alpha}\ (\Box\text{right})$$

$$\frac{\alpha \Rightarrow \Diamond\Gamma}{\Diamond\alpha \Rightarrow \Diamond\Gamma}\ (\Diamond\text{left}) \qquad \frac{\Gamma \Rightarrow \Delta, \alpha}{\Gamma \Rightarrow \Delta, \Diamond\alpha}\ (\Diamond\text{right}).$$

Some remarks concerning MML_n and S4C are given as follows.

1. The proposed system MML_n is regarded as a modal extension of the original system GML_n introduced in [30] and the related systems studied in [17].
2. The sequents of the form $\alpha \Rightarrow \alpha$ for any formula α are provable in MML_n and S4C. This fact can be shown by induction on α.
3. For any positive integers j and k with $j, k \leq n$ and $j \neq k$, the following sequents are provable in cut-free MML_n:
   (a) ${\sim_j}(\alpha{\rightarrow_j}\beta) \Leftrightarrow {\sim_j}\beta{\leftarrow_j}{\sim_j}\alpha$,
   (b) ${\sim_j}(\alpha{\leftarrow_j}\beta) \Leftrightarrow {\sim_j}\beta{\rightarrow_j}{\sim_j}\alpha$,
   (c) ${\sim_k}(\alpha{\rightarrow_j}\beta) \Leftrightarrow {\sim_k}\alpha{\rightarrow_j}{\sim_k}\beta$,
   (d) ${\sim_k}(\alpha{\leftarrow_j}\beta) \Leftrightarrow {\sim_k}\alpha{\leftarrow_j}{\sim_k}\beta$,
   (e) ${\sim_j}\Box_j\alpha \Leftrightarrow \Diamond_j{\sim_j}\alpha$,
   (f) ${\sim_j}\Diamond_j\alpha \Leftrightarrow \Box_j{\sim_j}\alpha$,
   (g) ${\sim_k}\Box_j\alpha \Leftrightarrow \Box_j{\sim_k}\alpha$,
   (h) ${\sim_k}\Diamond_j\alpha \Leftrightarrow \Diamond_j{\sim_k}\alpha$.
4. In [18,37] some Gentzen-type sequent calculi were studied, in which the following sequents are provable, similar to the sequents (a)–(d) above:
   (a) ${\sim}(\alpha{\rightarrow}\beta) \Leftrightarrow {\sim}\beta{\leftarrow}{\sim}\alpha$,
   (b) ${\sim}(\alpha{\leftarrow}\beta) \Leftrightarrow {\sim}\beta{\rightarrow}{\sim}\alpha$.
5. The $\{\Diamond, \leftarrow\}$-free part of S4C is the standard Gentzen-type sequent calculus for the normal modal logic S4 (see, e.g., [26]).
6. The co-implication connective $\leftarrow$ and the modal operator $\Diamond$ are standardly definable in S4, and hence, the systems S4C and S4 are logically equivalent.
7. It is well-known that the cut-elimination theorems holds for S4 (and hence, for S4C), and that S4 is decidable (and hence, so is S4C).
8. The rules ($\Box_j$right), (${\sim_j}\Diamond_j$right) and (${\sim_k}\Box_j$right), however cumbersome they may seem, are just generalizations of the standard S4C-rule ($\Box$right). Indeed, the following sequents are provable in MML_n:
   (a) ${\sim_j}\Diamond_j\alpha \Leftrightarrow \Box_j{\sim_j}\alpha$,
   (b) ${\sim_k}\Box_j\alpha \Leftrightarrow \Box_j{\sim_k}\alpha$.

   Hence, the context $\Box_j\Gamma, {\sim_j}\Diamond_j\Sigma, {\sim_k}\Box_j\Pi$ in these rules can be equivalently transformed into $\Box_j\Gamma, \Box_j{\sim_j}\Sigma, \Box_j{\sim_k}\Pi$, revealing thus its structure as a genuine generalization of $\Box_j\Gamma$ to formulas with paraconsistent negations. Analogous considerations apply to the rules ($\Diamond_j$left), ($\sim_j\Box_j$left) and ($\sim_k\Diamond_j$left), and their relationships to the S4C-rule ($\Diamond$left).

One can observe a beautiful *duality* between some inference rules in MML_n, as well as in S4C, obtained by inverting all the sequents involved in these rules. In this sense, the structural rules (we-left) and (we-right) are dual to each other, whereas (cut) is self-dual. A connective is considered to be self-dual, if there is a duality between its left and right logical rules. Such are rules for negation operators, both in MML_n and in S4C. Two different connectives are regarded as mutually dual, if and only if the right (left) logical rule for the first connective is dual to the left (right) logical rule for the second connective. In this way, $\wedge_j$ and $\vee_j$, $\Box_j$ and $\Diamond_j$ are dual to each other in MML_n, as well as $\wedge$ and $\vee$, $\Box$ and $\Diamond$ are in S4C.

Remarkably, implications and co-implications considered in this paper, both in MML_n and in S4C turn out to be *not* mutually dual. Indeed, a proper dualization of ($\rightarrow$left) and ($\rightarrow$right) in S4C results in the following rules, which are clearly different from ($\leftarrow$left) and ($\leftarrow$right):

$$\frac{\beta, \Gamma \Rightarrow \Delta, \alpha}{\alpha \mapsto \beta, \Gamma \Rightarrow \Delta}\ (\mapsto \text{left}) \qquad \frac{\Gamma \Rightarrow \Delta, \beta \quad \alpha, \Sigma \Rightarrow \Pi}{\Gamma, \Sigma \Rightarrow \Delta, \Pi, \alpha \mapsto \beta}\ (\mapsto \text{right}),$$

where $\mapsto$ stands for the connective *directly* dual to S4-implication. In contrast, the co-implication connective $\leftarrow$ of S4C above, governed by ($\leftarrow$left) and ($\leftarrow$right), is *not* the direct dual S4-implication, but the *converse* of $\mapsto$, i.e. the converse dual S4-implication. It is a kind of a subtraction-operator, an algebraic counterpart of which is a well-known operation of pseudo-difference. Analogous observations hold true for the connectives $\rightarrow_j$, $\leftarrow_j$, and their duals in MML_n.

There is, however, a widespread tradition in the literature, by dualizing implication to deal not with the "straight" dual-implication, but with its converse (subtraction), and we follow this tradition in the present paper. A reader may consult [35, pp. 1084–1085] and [31, footnote 4] for more details about this tradition, and the interrelations between implications, converse implications and their duals. We will summarize the duality issue for MML_n by a duality principle, discussed in Sect. 6 below.

Now, we define a translation function from formulas of MML_n into those of S4C, and by using this function, we will show in the next section several theorems for embedding MML_n into S4C.

Definition 3.3. Let n (>1) be the positive integer determined by n-lattice, and let j, k be any positive integers with $j, k \leq n$ and $j \neq k$. We fix a set Φ of propositional variables and define for every j the sets $\Phi^j := \{p^j \mid p \in \Phi\}$ of propositional variables. The language $\mathcal{L}_{\mathrm{MML}}$ of MML_n is defined using Φ, $\wedge_j, \vee_j, \rightarrow_j, \leftarrow_j$, $\Box_j, \Diamond_j$ and $\sim_j$. The language $\mathcal{L}_{\mathrm{S4C}}$ of S4C is defined using Φ, $\Phi^1, \ldots, \Phi^n$, $\wedge$, $\vee$, $\rightarrow$, $\leftarrow$, $\Box$, $\Diamond$ and $\neg$. A mapping f from $\mathcal{L}_{\mathrm{MML}}$ to $\mathcal{L}_{\mathrm{S4C}}$ is defined inductively by:

1. For any $p \in \Phi$, for any $j \leq n$, $f(p) := p$ and $f({\sim_j} p) := p^j \in \Phi^j$,
2. $f(\alpha {\wedge_j} \beta) := f(\alpha) \wedge f(\beta)$,
3. $f(\alpha {\vee_j} \beta) := f(\alpha) \vee f(\beta)$,
4. $f(\alpha {\rightarrow_j} \beta) := f(\alpha) {\rightarrow} f(\beta)$,
5. $f(\alpha {\leftarrow_j} \beta) := f(\alpha) {\leftarrow} f(\beta)$,
6. $f(\Box_j \alpha) := \Box f(\alpha)$,
7. $f(\Diamond_j \alpha) := \Diamond f(\alpha)$,
8. $f({\sim_j}(\alpha {\wedge_j} \beta)) := f({\sim_j}\alpha) \vee f({\sim_j}\beta)$,
9. $f({\sim_j}(\alpha {\vee_j} \beta)) := f({\sim_j}\alpha) \wedge f({\sim_j}\beta)$,
10. $f({\sim_j}(\alpha {\rightarrow_j} \beta)) := f({\sim_j}\beta) {\leftarrow} f({\sim_j}\alpha)$,
11. $f({\sim_j}(\alpha {\leftarrow_j} \beta)) := f({\sim_j}\beta) {\rightarrow} f({\sim_j}\alpha)$,
12. $f({\sim_j}\Box_j \alpha) := \Diamond f({\sim_j}\alpha)$,
13. $f({\sim_j}\Diamond_j \alpha) := \Box f({\sim_j}\alpha)$,
14. $f({\sim_j}{\sim_j}\alpha) := f(\alpha)$,
15. $f({\sim_k}(\alpha {\wedge_j} \beta)) := f({\sim_k}\alpha) \wedge f({\sim_k}\beta)$,
16. $f({\sim_k}(\alpha {\vee_j} \beta)) := f({\sim_k}\alpha) \vee f({\sim_k}\beta)$,
17. $f({\sim_k}(\alpha {\rightarrow_j} \beta)) := f({\sim_k}\alpha) {\rightarrow} f({\sim_k}\beta)$,
18. $f({\sim_k}(\alpha {\leftarrow_j} \beta)) := f({\sim_k}\alpha) {\leftarrow} f({\sim_k}\beta)$,
19. $f({\sim_k}\Box_j \alpha) := \Box f({\sim_k}\alpha)$,
20. $f({\sim_k}\Diamond_j \alpha) := \Diamond f({\sim_k}\alpha)$,
21. $f({\sim_k}{\sim_j}\alpha) := \neg f(\alpha)$.

Note that definition of f, as well the definition of the converse translational function g below essentially depends on the corresponding languages $\mathcal{L}_{\mathrm{MML}}$ and $\mathcal{L}_{\mathrm{S4C}}$, which are fixed with respect to a given multilattice. More concretely, having an initial set of propositional variables for $\mathcal{L}_{\mathrm{S4C}}$ and the integer n for the given $\mathcal{M}_n$ the corresponding sets of propositional variables Φ^j is easily definable for each $j \leq n$.

In what follows expression $f(\Gamma)$ denotes the result of replacing every occurrence of a formula α in Γ by $f(\alpha)$. Analogous notation is used for the other mappings discussed later.

Next, we introduce a translation function from formulas of S4C into those of MML_n, and by using this function, we will show in the latter section several theorems for embedding S4C into MML_n.

Definition 3.4. Let $\mathcal{L}_{\mathrm{MML}}$ and $\mathcal{L}_{\mathrm{S4C}}$ be the languages defined in Definition 3.3.

A mapping g from $\mathcal{L}_{\mathrm{S4C}}$ to $\mathcal{L}_{\mathrm{MML}}$ is defined inductively by:

1. For any $j \leq n$, any $p \in \Phi$, and any $p^j \in \Phi^j$, $g(p) := p$ and $g(p^j) := {\sim_j} p$;
2. $g(\alpha \wedge \beta) := g(\alpha) {\wedge_j} g(\beta)$, where j is a fixed positive integer, such that $j \leq n$;
3. $g(\alpha \vee \beta) := g(\alpha) {\vee_j} g(\beta)$, where j is a fixed positive integer, such that $j \leq n$;
4. $g(\alpha {\rightarrow} \beta) := g(\alpha) {\rightarrow_j} g(\beta)$, where j is a fixed positive integer, such that $j \leq n$;

5. $g(\alpha\leftarrow\beta) := g(\alpha)\leftarrow_j g(\beta)$, where j is a fixed positive integer, such that $j \leq n$;
6. $g(\neg\alpha) := \sim_k\sim_j g(\alpha)$, where j and k are two fixed positive integers, such that $j, k \leq n$ and $j \neq k$;
7. $g(\Box\alpha) := \Box_j g(\alpha)$, where j is a fixed positive integer, such that $j \leq n$;
8. $g(\Diamond\alpha) := \Diamond_j g(\alpha)$, where j is a fixed positive integer, such that $j \leq n$.

Some remarks on the above mentioned translation functions are given as follows.

1. The translation function f in Definition 3.3 is an extension of the translation function introduced in [17] for the non-modal fragment of MML_n.
2. A similar translation has been used by Gurevich [12], Rautenberg [29] and Vorob'ev [36] to embed Nelson's constructive logic [1,23] into intuitionistic logic.
3. Some similar translations have also been studied in [13–16] to embed some paraconsistent logics into classical logic.
4. The translation function g in Definition 3.4 is an extension of the translation function introduced in [17] for the non-modal fragment of S4C (i.e., a Gentzen-type sequent calculus LK for classical logic).

4. Syntactical Embedding and Cut-Elimination

Now, we can prove several theorems for syntactical embedding MML_n into S4C. Similar theorems were proved in [17] for the non-modal fragments of MML_n and S4C.

Theorem 4.1 (Weak syntactical embedding from MML_n into S4C). *Let* Γ, Δ *be sets of formulas in* $\mathcal{L}_{\mathrm{MML}}$*, and* f *be the mapping defined in Definition* 3.3*.*

1. *If* $\mathrm{MML}_n \vdash \Gamma \Rightarrow \Delta$*, then* $\mathrm{S4C} \vdash f(\Gamma) \Rightarrow f(\Delta)$*.*
2. *If* $\mathrm{S4C} - (\mathrm{cut}) \vdash f(\Gamma) \Rightarrow f(\Delta)$*, then* $\mathrm{MML}_n - (\mathrm{cut}) \vdash \Gamma \Rightarrow \Delta$*.*

Proof. • (1): By induction on the proofs P of $\Gamma \Rightarrow \Delta$ in MML_n. We distinguish the cases according to the last inference of P, and show some cases.

1. Case ($\sim_j p \Rightarrow \sim_j p$): The last inference of P is of the form: $\sim_j p \Rightarrow \sim_j p$ for any $p \in \Phi$. In this case, we obtain $\mathrm{S4C} \vdash f(\sim_j p) \Rightarrow f(\sim_j p)$, i.e., $\mathrm{S4C} \vdash p^j \Rightarrow p^j$ ($p^j \in \Phi^j$), by the definition of f.
2. Case ($\sim_k\sim_j$left): The last inference of P is of the form:

$$\frac{\Gamma \Rightarrow \Delta, \alpha}{\sim_k\sim_j\alpha, \Gamma \Rightarrow \Delta}\ (\sim_k\sim_j\text{left}).$$

By induction hypothesis, we have $\mathrm{S4C} \vdash f(\Gamma) \Rightarrow f(\Delta), f(\alpha)$. Then, we obtain the required fact:

$$\frac{\begin{array}{c}\vdots\\ f(\Gamma) \Rightarrow f(\Delta), f(\alpha)\end{array}}{\neg f(\alpha), f(\Gamma) \Rightarrow f(\Delta)}\ (\neg\text{left})$$

where $\neg f(\alpha)$ coincides with $f({\sim_k}{\sim_j}\alpha)$ by the definition of f.

3. Case ($\Box_j$right): The last inference of P is of the form:

$$\frac{\Box_j\Gamma, {\sim_j}\Diamond_j\Sigma, {\sim_k}\Box_j\Pi \Rightarrow \alpha}{\Box_j\Gamma, {\sim_j}\Diamond_j\Sigma, {\sim_k}\Box_j\Pi \Rightarrow \Box_j\alpha}\ (\Box_j\text{right}).$$

By induction hypothesis, we have S4C $\vdash f(\Box_j\Gamma), f({\sim_j}\Diamond_j\Sigma), f({\sim_k}\Box_j\Pi) \Rightarrow f(\alpha)$ where $f(\Box_j\Gamma)$, $f({\sim_j}\Diamond_j\Sigma)$ and $f({\sim_k}\Box_j\Pi)$ coincide with $\Box f(\Gamma)$, $\Box f({\sim_j}\Sigma)$ and $\Box f({\sim_k}\Pi)$, respectively, by the definition of f. Then, we obtain the required fact:

$$\frac{\begin{array}{c}\vdots\\ \Box f(\Gamma), \Box f({\sim_j}\Sigma), \Box f({\sim_k}\Pi) \Rightarrow f(\alpha)\end{array}}{\Box f(\Gamma), \Box f({\sim_j}\Sigma), \Box f({\sim_k}\Pi) \Rightarrow \Box f(\alpha)}\ (\Box\text{right})$$

where $\Box f(\alpha)$ coincides with $f(\Box_j\alpha)$ by the definition of f.

4. Case (${\sim_j}\Box_j$left): The last inference of P is of the form:

$$\frac{{\sim_j}\alpha \Rightarrow \Diamond_j\Gamma, {\sim_j}\Box_j\Sigma, {\sim_k}\Diamond_j\Pi}{{\sim_j}\Box_j\alpha \Rightarrow \Diamond_j\Gamma, {\sim_j}\Box_j\Sigma, {\sim_k}\Diamond_j\Pi}\ ({\sim_j}\Box_j\text{left}).$$

By induction hypothesis, we have S4C $\vdash f(\Box_j\alpha) \Rightarrow f(\Diamond_j\Gamma), f({\sim_j}\Box_j\Sigma), f({\sim_k}\Diamond_j\Pi)$ where $f(\Diamond_j\Gamma)$, $f({\sim_j}\Box_j\Sigma)$ and $f({\sim_k}\Diamond_j\Pi)$ coincide with $\Diamond f(\Gamma)$, $\Diamond f({\sim_j}\Sigma)$ and $\Diamond f({\sim_k}\Pi)$, respectively, by the definition of f. Then, we obtain the required fact:

$$\frac{\begin{array}{c}\vdots\\ f({\sim_j}\alpha) \Rightarrow \Diamond f(\Gamma), \Diamond f({\sim_j}\Sigma), \Diamond f({\sim_k}\Pi)\end{array}}{\Diamond f({\sim_j}\alpha) \Rightarrow \Diamond f(\Gamma), \Diamond f({\sim_j}\Sigma), \Diamond f({\sim_k}\Pi)}\ (\Diamond\text{left})$$

where $\Diamond f({\sim_j}\alpha)$ coincides with $f({\sim_j}\Box_j\alpha)$ by the definition of f.

5. Case (${\sim_k}\Box_j$right): The last inference of P is of the form:

$$\frac{\Box_j\Gamma, {\sim_j}\Diamond_j\Sigma, {\sim_k}\Box_j\Pi \Rightarrow {\sim_k}\alpha}{\Box_j\Gamma, {\sim_j}\Diamond_j\Sigma, {\sim_k}\Box_j\Pi \Rightarrow {\sim_k}\Box_j\alpha}\ ({\sim_k}\Box_j\text{right}).$$

By induction hypothesis, we have S4C $\vdash f(\Box_j\Gamma), f({\sim_j}\Diamond_j\Sigma), f({\sim_k}\Box_j\Pi) \Rightarrow f({\sim_k}\alpha)$ where $f(\Box_j\Gamma)$, $f({\sim_j}\Diamond_j\Sigma)$ and $f({\sim_k}\Box_j\Pi)$ coincide with $\Box f(\Gamma)$, $\Box f({\sim_j}\Sigma)$ and $\Box f({\sim_k}\Pi)$, respectively, by the definition of f. Then, we obtain the required fact:

$$\frac{\begin{array}{c}\vdots\\ \Box f(\Gamma), \Box f({\sim_j}\Sigma), \Box f({\sim_k}\Pi) \Rightarrow f({\sim_k}\alpha)\end{array}}{\Box f(\Gamma), \Box f({\sim_j}\Sigma), \Box f({\sim_k}\Pi) \Rightarrow \Box f({\sim_k}\alpha)}\ (\Box\text{right})$$

where $\Box f({\sim_k}\alpha)$ coincides with $f({\sim_k}\Box_j\alpha)$ by the definition of f.

• (2): By induction on the proofs Q of $f(\Gamma) \Rightarrow f(\Delta)$ in S4C − (cut). We distinguish the cases according to the last inference of Q, and show only the following cases.

1. Case ($\neg$left): The last inference of Q is of the form:

$$\frac{f(\Gamma) \Rightarrow f(\Delta), f(\alpha)}{f({\sim_k}{\sim_j}\alpha), f(\Gamma) \Rightarrow f(\Delta)}\ (\neg\text{left})$$

where $f({\sim_k}{\sim_j}\alpha)$ coincides with $\neg f(\alpha)$ by the definition of f. By induction hypothesis, we have MML_n − (cut) $\vdash \Gamma \Rightarrow \Delta, \alpha$. We thus obtain the required fact:

$$\frac{\begin{array}{c}\vdots\\ \Gamma \Rightarrow \Delta, \alpha\end{array}}{{\sim_k}{\sim_j}\alpha, \Gamma \Rightarrow \Delta}\ ({\sim_k}{\sim_j}\text{left}).$$

2. Case ($\Box$right): The last inference of Q is ($\Box$right).
   (a) Subcase (1): The last inference of Q is of the form:

$$\frac{f(\Box_j\Gamma), f({\sim_j}\Diamond_j\Sigma), f({\sim_k}\Box_j\Pi) \Rightarrow f(\alpha)}{f(\Box_j\Gamma), f({\sim_j}\Diamond_j\Sigma), f({\sim_k}\Box_j\Pi) \Rightarrow f(\Box_j\alpha)}\ (\Box\text{right})$$

where $f(\Box_j\Gamma)$, $f({\sim_j}\Diamond_j\Sigma)$, $f({\sim_k}\Box_j\Pi)$ and $f(\Box_j\alpha)$ coincide with $\Box f(\Gamma)$, $\Box f({\sim_j}\Sigma)$, $\Box f({\sim_k}\Pi)$ and $\Box f(\alpha)$, respectively, by the definition of f. By induction hypothesis, we have MML_n − (cut) $\vdash$ $\Box_j\Gamma, {\sim_j}\Diamond_j\Sigma, {\sim_k}\Box_j\Pi \Rightarrow \alpha$. We thus obtain the required fact:

$$\frac{\begin{array}{c}\vdots\\ \Box_j\Gamma, {\sim_j}\Diamond_j\Sigma, {\sim_k}\Box_j\Pi \Rightarrow \alpha\end{array}}{\Box_j\Gamma, {\sim_j}\Diamond_j\Sigma, {\sim_k}\Box_j\Pi \Rightarrow \Box_j\alpha}\ (\Box_j\text{right}).$$

   (b) Subcase (2): The last inference of Q is of the form:

$$\frac{f(\Box_j\Gamma), f({\sim_j}\Diamond_j\Sigma), f({\sim_k}\Box_j\Pi) \Rightarrow f({\sim_j}\alpha)}{f(\Box_j\Gamma), f({\sim_j}\Diamond_j\Sigma), f({\sim_k}\Box_j\Pi) \Rightarrow f({\sim_j}\Diamond_j\alpha)}\ (\Box\text{right})$$

where $f(\Box_j\Gamma)$, $f({\sim_j}\Diamond_j\Sigma)$, $f({\sim_k}\Box_j\Pi)$ and $f({\sim_j}\Diamond_j\alpha)$ coincide with $\Box f(\Gamma)$, $\Box f({\sim_j}\Sigma)$, $\Box f({\sim_k}\Pi)$ and $\Box f({\sim_j}\alpha)$, respectively, by the definition of f. By induction hypothesis, we have MML_n − (cut) $\vdash$ $\Box_j\Gamma, {\sim_j}\Diamond_j\Sigma, {\sim_k}\Box_j\Pi \Rightarrow {\sim_j}\alpha$. We thus obtain the required fact:

$$\frac{\begin{array}{c}\vdots\\ \Box_j\Gamma, {\sim_j}\Diamond_j\Sigma, {\sim_k}\Box_j\Pi \Rightarrow {\sim_j}\alpha\end{array}}{\Box_j\Gamma, {\sim_j}\Diamond_j\Sigma, {\sim_k}\Box_j\Pi \Rightarrow {\sim_j}\Diamond_j\alpha}\ ({\sim_j}\Diamond_j\text{right}).$$

   (c) Subcase (3): The last inference of Q is of the form:

$$\frac{f(\Box_j\Gamma), f({\sim_j}\Diamond_j\Sigma), f({\sim_k}\Box_j\Pi) \Rightarrow f({\sim_k}\alpha)}{f(\Box_j\Gamma), f({\sim_j}\Diamond_j\Sigma), f({\sim_k}\Box_j\Pi) \Rightarrow f({\sim_k}\Box_j\alpha)}\ (\Box\text{right})$$

where $f(\Box_j\Gamma)$, $f({\sim_j}\Diamond_j\Sigma)$, $f({\sim_k}\Box_j\Pi)$ and $f({\sim_k}\Box_j\alpha)$ coincide with $\Box f(\Gamma)$, $\Box f({\sim_j}\Sigma)$, $\Box f({\sim_k}\Pi)$ and $\Box f({\sim_k}\alpha)$, respectively, by the definition of f. By induction hypothesis, we have MML_n − (cut) $\vdash$

$\Box_j\Gamma, {\sim_j}\Diamond_j\Sigma, {\sim_k}\Box_j\Pi \Rightarrow {\sim_k}\alpha$. We thus obtain the required fact:

$$\dfrac{\begin{array}{c}\vdots\\ \Box_j\Gamma, {\sim_j}\Diamond_j\Sigma, {\sim_k}\Box_j\Pi \Rightarrow {\sim_k}\alpha\end{array}}{\Box_j\Gamma, {\sim_j}\Diamond_j\Sigma, {\sim_k}\Box_j\Pi \Rightarrow {\sim_k}\Box_j\alpha}\ ({\sim_k}\Box_j\text{right}).$$

□

Theorem 4.2 (Cut-elimination for MML_n). *The rule* (cut) *is admissible in cut-free MML_n.*

Proof. Suppose $\mathrm{MML}_n \vdash \Gamma \Rightarrow \Delta$. Then, we have S4C $\vdash f(\Gamma) \Rightarrow f(\Delta)$ by Theorem 4.1 (1), and hence S4C − (cut) $\vdash f(\Gamma) \Rightarrow f(\Delta)$ by the cut-elimination theorem for S4C. By Theorem 4.1 (2), we obtain MML_n − (cut) $\vdash \Gamma \Rightarrow \Delta$. □

Theorem 4.3 (Syntactical embedding from MML_ninto S4C). *Let Γ, Δ be sets of formulas in $\mathcal{L}_{\mathrm{MML}}$, and f be the mapping defined in Definition 3.3.*

1. *$MML_n \vdash \Gamma \Rightarrow \Delta$ iff* S4C $\vdash f(\Gamma) \Rightarrow f(\Delta)$.
2. *MML_n* − (cut) $\vdash \Gamma \Rightarrow \Delta$ *iff* S4C − (cut) $\vdash f(\Gamma) \Rightarrow f(\Delta)$.

Proof. • (1): ($\Longrightarrow$): By Theorem 4.1 (1). ($\Longleftarrow$): Suppose S4C $\vdash f(\Gamma) \Rightarrow f(\Delta)$. Then we have S4C − (cut) $\vdash f(\Gamma) \Rightarrow f(\Delta)$ by the cut-elimination theorem for S4C. We thus obtain MML_n − (cut) $\vdash \Gamma \Rightarrow \Delta$ by Theorem 4.1 (2). Therefore we have $\mathrm{MML}_n \vdash \Gamma \Rightarrow \Delta$.

• (2): ($\Longrightarrow$): Suppose MML_n − (cut) $\vdash \Gamma \Rightarrow \Delta$. Then we have $\mathrm{MML}_n \vdash \Gamma \Rightarrow \Delta$. We then obtain S4C $\vdash f(\Gamma) \Rightarrow f(\Delta)$ by Theorem 4.1 (1). Therefore we obtain S4C − (cut) $\vdash f(\Gamma) \Rightarrow f(\Delta)$ by the cut-elimination theorem for S4C. ($\Longleftarrow$): By Theorem 4.1 (2). □

Theorem 4.4 (Decidability for MML_n). *MML_n is decidable.*

Proof. By decidability of S4C, for each α, it is possible to decide if $\Rightarrow f(\alpha)$ is provable in S4C. Then, by Theorem 4.3, MML_n is also decidable. □

An expression $V(\alpha)$ denotes the set of all propositional variables occurring in α. By using Theorem 4.3, we can also obtain the following theorem for MML_n. This theorem can be proved as a straightforward extension of the proof of the same theorem in [17].

Theorem 4.5 (Modified Craig interpolation for MML_n). *Let n (>1) be the positive integer determined by n-lattice, and let j, k be any positive integers with $j, k \leq n$ and $j \neq k$. Suppose $MML_n \vdash \alpha \Rightarrow \beta$ for any formulas α and β. If $V(\alpha) \cap V(\beta) \neq \emptyset$, then there exists a formula γ such that*

1. *$MML_n \vdash \alpha \Rightarrow \gamma$ and $MML_n \vdash \gamma \Rightarrow \beta$,*
2. *$V(\gamma) \subseteq V(\alpha) \cap V(\beta)$.*

If $V(\alpha) \cap V(\beta) = \emptyset$, then

3. *$MML_n \vdash\ \Rightarrow {\sim_k}{\sim_j}\alpha$ or $MML_n \vdash\ \Rightarrow \beta$.*

By using Theorem 4.5, we can obtain the following theorem, which was also discussed in [17].

Theorem 4.6 (Maksimova's separation for MML_n). *Let n (>1) be the positive integer determined by n-lattice, and let i be any positive integer with $i \le n$. Suppose $V(\alpha_1, \alpha_2) \cap V(\beta_1, \beta_2) \neq \emptyset$ for any formulas $\alpha_1, \alpha_2, \beta_1$ and β_2. If $MML_n \vdash \alpha_1 \wedge_i \beta_1 \Rightarrow \alpha_2 \vee_i \beta_2$, then either $MML_n \vdash \alpha_1 \Rightarrow \alpha_2$ or $MML_n \vdash \beta_1 \Rightarrow \beta_2$.*

Next, we show some theorems for syntactical embedding S4C into MML_n. A similar theorem was shown in [17] for the non-modal fragments of MML_n and S4C.

Theorem 4.7 (Weak syntactical embedding from S4C into MML_n). *Let Γ, Δ be sets of formulas in $\mathcal{L}_{\mathrm{S4C}}$, and g be the mapping defined in Definition 3.4.*

1. *If* S4C $\vdash \Gamma \Rightarrow \Delta$*, then $MML_n \vdash g(\Gamma) \Rightarrow g(\Delta)$.*
2. *If MML_n $-$ (cut) $\vdash g(\Gamma) \Rightarrow g(\Delta)$, then* S4C $-$ (cut) $\vdash \Gamma \Rightarrow \Delta$.

Proof. • (1): By induction on the proofs P of $\Gamma \Rightarrow \Delta$ in S4C. We distinguish the cases according to the last inference of P, and show only the following cases.

1. Case ($p^j \Rightarrow p^j$): The last inference of P is of the form: $p^j \Rightarrow p^j$ for any $p^j \in \Phi^j$. In this case, we obtain $\mathrm{MML}_n \vdash g(p^j) \Rightarrow g(p^j)$, i.e., $\mathrm{MML}_n \vdash {\sim_j} p \Rightarrow {\sim_j} p$, by the definition of g.
2. Case ($\neg$left): The last inference of P is of the form:

$$\dfrac{\Gamma \Rightarrow \Delta, \alpha}{\neg\alpha, \Gamma \Rightarrow \Delta}\ (\neg\text{left})$$

   By induction hypothesis, we have $\mathrm{MML}_n \vdash g(\Gamma) \Rightarrow g(\Delta), g(\alpha)$. We then obtain the required fact:

$$\dfrac{\begin{array}{c}\vdots\\ f(\Gamma) \Rightarrow f(\Delta), g(\alpha)\end{array}}{{\sim_k}{\sim_j} g(\alpha), g(\Gamma) \Rightarrow f(\Delta)}\ ({\sim_k}{\sim_j}\text{left})$$

   where ${\sim_k}{\sim_j} g(\alpha)$ coincides with $g(\neg\alpha)$ by the definition of g.
3. Case ($\Box$right): The last inference of P is of the form:

$$\dfrac{\Box\Gamma \Rightarrow \alpha}{\Box\Gamma \Rightarrow \Box\alpha}\ (\Box\text{right})$$

   By induction hypothesis, we have $\mathrm{MML}_n \vdash g(\Box\Gamma) \Rightarrow g(\alpha)$ where $g(\Box\Gamma)$ coincides with $\Box_j g(\Gamma)$ by the definition of g. We then obtain the required fact:

$$\dfrac{\begin{array}{c}\vdots\\ \Box_j g(\Gamma) \Rightarrow g(\alpha)\end{array}}{\Box_j g(\Gamma) \Rightarrow \Box_j g(\alpha)}\ (\Box_j\text{right})$$

   where $\Box_j g(\alpha)$ coincides with $g(\Box\alpha)$ by the definition of g.

• (2): By induction on the proofs Q of $g(\Gamma) \Rightarrow g(\Delta)$ in MML_n $-$ (cut). We distinguish the cases according to the last inference of Q, and show only the following cases.

1. Case ($\sim_k\sim_j$left): The last inference of Q is of the form:

$$\frac{g(\Gamma) \Rightarrow g(\Delta), g(\alpha)}{\sim_k\sim_j g(\alpha), g(\Gamma) \Rightarrow g(\Delta)}\ (\sim_k\sim_j\text{left})$$

   where $\sim_k\sim_j g(\alpha)$ coincides with $g(\neg\alpha)$ by the definition of g. By induction hypothesis, we have S4C $-$ (cut) $\vdash \Gamma \Rightarrow \Delta, \alpha$. We thus obtain the required fact:

$$\frac{\begin{array}{c}\vdots\\ \Gamma \Rightarrow \Delta, \alpha\end{array}}{\neg\alpha, \Gamma \Rightarrow \Delta}\ (\neg\text{left}).$$

2. Case ($\Box_j$right): The last inference of Q is of the form:

$$\frac{g(\Box\Gamma) \Rightarrow g(\alpha)}{g(\Box\Gamma) \Rightarrow \Box_j g(\alpha)}\ (\Box_j\text{right})$$

   where $g(\Box\Gamma)$ and $\Box_j g(\alpha)$ coincide with $\Box_j g(\Gamma)$ and $g(\Box\alpha)$, respectively, by the definition of g. By induction hypothesis, we have S4C $-$ (cut) $\vdash$ $\Box\Gamma \Rightarrow \alpha$. We thus obtain the required fact:

$$\frac{\begin{array}{c}\vdots\\ \Box\Gamma \Rightarrow \alpha\end{array}}{\Box\Gamma \Rightarrow \Box\alpha}\ (\Box\text{right}).$$

□

Theorem 4.8 (Syntactical embedding from S4C into MML$_n$). *Let* Γ, Δ *be sets of formulas in* $\mathcal{L}_{\text{S4C}}$, *and* g *be the mapping defined in Definition* 3.4.

1. S4C $\vdash \Gamma \Rightarrow \Delta$ *iff MML*$_n$ $\vdash g(\Gamma) \Rightarrow g(\Delta)$.
2. S4C $-$ (cut) $\vdash \Gamma \Rightarrow \Delta$ *iff MML*$_n$ $-$ (cut) $\vdash g(\Gamma) \Rightarrow g(\Delta)$.

Proof. By using Theorems 4.7 and 4.2. Similar to Theorem 4.3. □

5. Semantical Embedding and Completeness

We define a Kripke semantics for MML$_n$.

Definition 5.1. A structure $\langle M, R\rangle$ is called a *Kripke frame* if

1. M is a non-empty set,
2. R is a transitive and reflexive binary relation on M.

Definition 5.2. Let n (>1) be the positive integer determined by n-lattice, and j, k be any positive integers with $j, k \leq n$ and $j \neq k$. A *paraconsistent valuation* $\models^p$ on a Kripke frame $\langle M, R\rangle$ is a mapping from the set $\Phi \cup \Phi^{\sim}$ of all propositional variables and negated propositional variables to the power set 2^M of M. We will write $x \models^p p$ for $x \in \models^p (p)$. The paraconsistent valuation $\models^p$ is extended to the mapping from the set of all formulas to 2^M by:

1. $x \models^p \alpha{\wedge_j}\beta$ iff $x \models^p \alpha$ and $x \models^p \beta$,
2. $x \models^p \alpha{\vee_j}\beta$ iff $x \models^p \alpha$ or $x \models^p \beta$,
3. $x \models^p \alpha{\rightarrow_j}\beta$ iff $x \not\models^p \alpha$ or $x \models^p \beta$,

4. $x \models^p \alpha \leftarrow_j \beta$ iff $x \models^p \alpha$ and $x \not\models^p \beta$,
5. $x \models^p \Box_j \alpha$ iff $\forall y \in M$ $[xRy$ implies $y \models^p \alpha]$,
6. $x \models^p \Diamond_j \alpha$ iff $\exists y \in M$ $[xRy$ and $y \models^p \alpha]$,
7. $x \models^p \sim_j (\alpha \wedge_j \beta)$ iff $x \models^p \sim_j \alpha$ or $x \models^p \sim_j \beta$,
8. $x \models^p \sim_j (\alpha \vee_j \beta)$ iff $x \models^p \sim_j \alpha$ and $x \models^p \sim_j \beta$,
9. $x \models^p \sim_j (\alpha \rightarrow_j \beta)$ iff $x \models^p \sim_j \beta$ and $x \not\models^p \sim_j \alpha$,
10. $x \models^p \sim_j (\alpha \leftarrow_j \beta)$ iff $x \models^p \sim_j \beta$ implies $x \models^p \sim_j \alpha$,
11. $x \models^p \sim_j \Box_j \alpha$ iff $\exists y \in M$ $[xRy$ and $y \models^p \sim_j \alpha]$,
12. $x \models^p \sim_j \Diamond_j \alpha$ iff $\forall y \in M$ $[xRy$ implies $y \models^p \sim_j \alpha]$,
13. $x \models^p \sim_j \sim_j \alpha$ iff $x \models^p \alpha$,
14. $x \models^p \sim_k (\alpha \wedge_j \beta)$ iff $x \models^p \sim_k \alpha$ and $x \models^p \sim_k \beta$,
15. $x \models^p \sim_k (\alpha \vee_j \beta)$ iff $x \models^p \sim_k \alpha$ or $x \models^p \sim_k \beta$,
16. $x \models^p \sim_k (\alpha \rightarrow_j \beta)$ iff $x \models^p \sim_k \alpha$ implies $x \models^p \sim_k \beta$,
17. $x \models^p \sim_k (\alpha \leftarrow_j \beta)$ iff $x \models^p \sim_k \alpha$ and $x \not\models^p \sim_k \beta$,
18. $x \models^p \sim_k \Box_j \alpha$ iff $\forall y \in M$ $[xRy$ implies $y \models^p \sim_k \alpha]$,
19. $x \models^p \sim_k \Diamond_j \alpha$ iff $\exists y \in M$ $[xRy$ and $y \models^p \sim_k \alpha]$,
20. $x \models^p \sim_k \sim_j \alpha$ iff $x \not\models^p \alpha$.

Incidentally, clauses 3 and 4 of this definition show that $\leftarrow_j$ is the *converse* dual connective to $\rightarrow_j$. Indeed, the truth condition for the formula dual to $\alpha \rightarrow_j \beta$ should be $x \not\models^p \alpha$ and $x \models^p \beta$, obtained by a direct dualization of the truth condition for $\rightarrow_j$.

It is also noteworthy, that truth conditions for modal operators are modeled here through a single (unified) accessibility relation, in contrast to a widely accepted approach in many-valued modal logics, where each operator is usually equipped with its own accessibility relation. Such a definition anticipates a possibility of a mutual translation between multilattice and usual modal logics.

Definition 5.3. A *paraconsistent Kripke model* is a structure $\langle M, R, \models^p \rangle$, such that

1. $\langle M, R \rangle$ is a Kripke frame,
2. $\models^p$ is a paraconsistent valuation on $\langle M, R \rangle$.

A formula α is *true* in a paraconsistent Kripke model $\langle M, R, \models^p \rangle$ iff $x \models^p \alpha$ for any $x \in M$, and is *MML_n-valid* in a Kripke frame $\langle M, R \rangle$ iff it is true for every paraconsistent valuation $\models^p$ on the Kripke frame.

In order to show a theorem for semantical embedding MML_n into S4C, we present the standard Kripke semantics for S4C.

Definition 5.4. A *valuation* $\models$ on a Kripke frame $\langle M, R \rangle$ is a mapping from the set Φ of all propositional variables to the power set 2^M of M. We will write $x \models p$ for $x \in \models (p)$. The valuation $\models$ is extended to a mapping from the set of all formulas to 2^M by:

1. $x \models \alpha \wedge \beta$ iff $x \models \alpha$ and $x \models \beta$,
2. $x \models \alpha \vee \beta$ iff $x \models \alpha$ or $x \models \beta$,
3. $x \models \alpha \rightarrow \beta$ iff $x \not\models \alpha$ or $x \models \beta$,
4. $x \models \alpha \leftarrow \beta$ iff $x \models \alpha$ and $x \not\models \beta$,
5. $x \models \neg \alpha$ iff $x \not\models \alpha$,

6. $x \models \Box\alpha$ iff $\forall y \in M$ $[xRy$ implies $y \models \alpha]$,
7. $x \models \Diamond\alpha$ iff $\exists y \in M$ $[xRy$ and $y \models \alpha]$.

Again, clauses 3 and 4 of this definition show that $\leftarrow$ is the converse dual connective to $\rightarrow$. Indeed, the truth condition for the formula dual to $\alpha \rightarrow \beta$ should be $x \not\models \alpha$ and $x \models \beta$, obtained by a direct dualization of the truth condition for $\rightarrow$.

Definition 5.5. A *Kripke model* is a structure $\langle M, R, \models\rangle$ such that

1. $\langle M, R\rangle$ is a Kripke frame,
2. $\models$ is a valuation on $\langle M, R\rangle$.

A formula α is *true* in a Kripke model $\langle M, R, \models\rangle$ iff $x \models \alpha$ for any $x \in M$, and is *S4C-valid* in a Kripke frame $\langle M, R\rangle$ iff it is true for every valuation $\models$ on the Kripke frame.

The following completeness theorem for S4C is known. For any formula α,

S4C $\vdash \Rightarrow \alpha$ iff α is S4C-valid.

Next, we show a theorem for semantical embedding MML_n into S4C, and by using this theorem, we show the Kripke completeness theorem for MML_n. Prior to prove the semantical embedding theorem, we need to show some lemmas.

Lemma 5.6. *Let f be the mapping defined in Definition 3.3. For any paraconsistent Kripke model $\langle M, R, \models^p\rangle$, we can construct a Kripke model $\langle M, R, \models\rangle$ such that for any formula α and any $x \in M$,*

$x \models^p \alpha$ *iff* $x \models f(\alpha)$.

Proof. Let Φ be a set of propositional variables, $\Phi^{\sim}$ be the set $\{\sim_j p \mid p \in \Phi$ & $1 \leq j \leq n\}$ of negated propositional variables, and Φ^j $(1 \leq j \leq n)$ be the sets $\{p^j \mid p \in \Phi\}$ of propositional variables. Suppose that $\langle M, R, \models^p\rangle$ is a paraconsistent Kripke model where $\models^p$ is a mapping from $\Phi \cup \Phi^{\sim}$ to 2^M. Suppose that $\langle M, R, \models\rangle$ is a Kripke model where $\models$ is a mapping from $\Phi \cup \bigcup_{1\leq j\leq n} \Phi^j$ to 2^M such that for any $x \in M$ and any $p \in \Phi$,

1. $x \models^p p$ iff $x \models p$,
2. $x \models^p \sim_j p$ iff $x \models p^j$.

Then, the lemma is proved by induction on α.

• Base step:

1. Case when α is p, where p is a propositional variable: $x \models^p p$ iff $x \models p$ (by the assumption) iff $x \models f(p)$ (by the definition of f).
2. Case when α is $\sim_j p$, where p is a propositional variable: $x \models^p \sim_j p$ iff $x \models p^j$ (by the assumption) iff $x \models f(\sim_j p)$ (by the definition of f).

• Induction step: We show some cases.

1. Case when α is $\beta\wedge_j\gamma$: $x \models^p \beta\wedge_j\gamma$ iff $x \models^p \beta$ and $x \models^p \gamma$ iff $x \models f(\beta)$ and $x \models f(\gamma)$ (by induction hypothesis) iff $x \models f(\beta) \wedge f(\gamma)$ iff $x \models f(\beta\wedge_j\gamma)$ (by the definition of f).

2. Case when α is $\beta{\rightarrow_j}\gamma$: $x \models^p \beta{\rightarrow_j}\gamma$ iff $x \models^p \beta$ implies $x \models^p \gamma$ iff $x \models f(\beta)$ implies $x \models f(\gamma)$ (by induction hypothesis) iff $x \models f(\beta){\rightarrow}f(\gamma)$ iff $x \models f(\beta{\rightarrow_j}\gamma)$ (by the definition of f).
3. Case when α is $\Box_j\beta$: $x \models^p \Box_j\beta$ iff $\forall y \in M[xRy$ implies $y \models^p \beta]$ iff $\forall y \in M[xRy$ implies $y \models f(\beta)]$ (by induction hypothesis) iff $x \models \Box f(\beta)$ iff $x \models f(\Box_j\beta)$ (by the definition of f).
4. Case when α is $\sim_j\sim_j\beta$: $x \models^p \sim_j\sim_j\beta$ iff $x \models^p \beta$ iff $x \models f(\beta)$ (by induction hypothesis) iff $x \models f(\sim_j\sim_j\beta)$ (by the definition of f).
5. Case when α is $\sim_j(\beta\wedge_j\gamma)$: $x \models^p \sim_j(\beta\wedge_j\gamma)$ iff $x \models^p \sim_j\beta$ or $x \models^p \sim_j\gamma$ iff $x \models f(\sim_j\beta)$ or $x \models f(\sim_j\gamma)$ (by induction hypothesis) iff $x \models f(\sim_j\beta) \vee f(\sim_j\gamma)$ iff $x \models f(\sim_j(\beta\wedge_j\gamma))$ (by the definition of f).
6. Case when α is $\sim_j(\beta{\rightarrow_j}\gamma)$: $x \models^p \sim_j(\beta{\rightarrow_j}\gamma)$ iff $x \models^p \sim_j\gamma$ and $x \not\models^p \sim_j\beta$ iff $x \models f(\sim_j\gamma)$ and $x \not\models f(\sim_j\beta)$ (by induction hypotheses) iff $x \models f(\sim_j\gamma){\leftarrow}f(\sim_j\beta)$ iff $x \models f(\sim_j(\beta{\rightarrow_j}\gamma))$ (by the definition of f).
7. Case when α is $\sim_j\Box_j\beta$: $x \models^p \sim_j\Box_j\beta$ iff $\exists y \in M[xRy$ and $y \models^p \sim_j\beta]$ iff $\exists y \in M[xRy$ and $y \models f(\sim_j\beta)]$ (by induction hypothesis) iff $x \models \Diamond f(\sim_j\beta)$ iff $x \models f(\sim_j\Box_j\beta)$ (by the definition of f).
8. Case when α is $\sim_k\sim_j\beta$: $x \models^p \sim_k\sim_j\beta$ iff $x \not\models^p (\beta)$ iff $x \not\models f(\beta)$ (by induction hypothesis) iff $x \models \neg f(\beta)$ iff $x \models f(\sim_k\sim_j\beta)$ (by the definition of f).
9. Case when α is $\sim_k(\beta\wedge_j\gamma)$: $x \models^p \sim_k(\beta\wedge_j\gamma)$ iff $x \models^p \sim_k\beta$ and $x \models^p \sim_k\gamma$ iff $x \models f(\sim_k\beta)$ and $x \models f(\sim_k\gamma)$ (by induction hypothesis) iff $x \models f(\sim_k\beta) \wedge f(\sim_k\gamma)$ iff $x \models f(\sim_k(\beta\wedge_j\gamma))$ (by the definition of f).
10. Case when α is $\sim_k(\beta{\rightarrow_j}\gamma)$: $x \models^p \sim_k(\beta{\rightarrow_j}\gamma)$ iff $x \models^p \sim_k\beta$ implies $x \models^p \sim_k\gamma$ iff $x \models f(\sim_k\beta)$ implies $x \models f(\sim_k\gamma)$ (by induction hypothesis) iff $x \models f(\sim_k\beta){\rightarrow}f(\sim_k\gamma)$ iff $x \models f(\sim_k(\beta{\rightarrow_j}\gamma))$ (by the definition of f).
11. Case when α is $\sim_k\Box_j\beta$: $x \models^p \sim_k\Box_j\beta$ iff $\forall y \in M[xRy$ implies $y \models^p \sim_k\beta]$ iff $\forall y \in M[xRy$ implies $y \models f(\sim_k\beta)]$ (by induction hypothesis) iff $x \models \Box f(\sim_k\beta)$ iff $x \models f(\sim_k\Box_j\beta)$ (by the definition of f).

□

Lemma 5.7. *Let f be the mapping defined in Definition 3.3. For any Kripke model $\langle M, R, \models\rangle$, we can construct a paraconsistent Kripke model $\langle M, R, \models^p\rangle$ such that for any formula α and any $x \in M$,*

$$x \models f(\alpha) \text{ iff } x \models^p \alpha.$$

Proof. Similar to the proof of Lemma 5.6. □

Theorem 5.8 (Semantical embedding from MML_ninto S4C). *Let f be the mapping defined in Definition 3.3. For any formula α,*

$$\alpha \text{ is } MML_n\text{-valid iff } f(\alpha) \text{ is } S4C\text{-valid}.$$

Proof. By Lemmas 5.6 and 5.7. □

Theorem 5.9 (Completeness for MML_n). *For any formula α,*

$$\mathrm{MML}_n \vdash \Rightarrow \alpha \text{ iff } \alpha \text{ is } MML_n\text{-valid}.$$

Proof. We have the following. $\mathrm{MML}_n \vdash \Rightarrow \alpha$ iff $\mathrm{S4C} \vdash \Rightarrow f(\alpha)$ (by Theorem 4.3) iff $f(\alpha)$ is S4C-valid (by the completeness theorem for S4C) iff α is MML_n-valid (by Theorem 5.8). □

Next, we show a theorem for semantical embedding S4C into MML_n.

Lemma 5.10. *Let g be the mapping defined in Definition 3.4. For any Kripke model $\langle M, R, \models\rangle$, we can construct a paraconsistent Kripke model $\langle M, R, \models^p\rangle$ such that for any formula α and any $x \in M$,*

$x \models \alpha$ *iff* $x \models^p g(\alpha)$.

Proof. Let Φ be a set of propositional variables, $\Phi^{\sim}$ be the set $\{\sim_j p \mid p \in \Phi$ & $1 \leq j \leq n\}$ of negated propositional variables, and Φ^j $(1 \leq j \leq n)$ be the sets $\{p^j \mid p \in \Phi\}$ of propositional variables. Suppose that $\langle M, R, \models\rangle$ is a Kripke model where $\models$ is a mapping from $\Phi \cup \bigcup_{1\leq j\leq n} \Phi^j$ to 2^M. Suppose that $\langle M, R, \models^p\rangle$ is a paraconsistent Kripke model where $\models^p$ is a mapping from $\Phi \cup \Phi^{\sim}$ to 2^M such that for any $x \in M$ and any $p \in \Phi$,

1. $x \models^p p$ iff $x \models p$,
2. $x \models^p \sim_j p$ iff $x \models p^j$.

Then, the lemma is proved by induction on α.

• Base step:

1. Case when α is p, where p is a propositional variable: $x \models p$ iff $x \models^p p$ (by the assumption) iff $x \models^p g(p)$ (by the definition of g).
2. Case when α is p^j, where p is a propositional variable: $x \models p^j$ iff $x \models^p \sim_j p$ (by the assumption) iff $x \models^p g(p^j)$ (by the definition of g).

• Induction step: We show some cases.

1. Case when α is $\beta \wedge \gamma$: $x \models \beta \wedge \gamma$ iff $x \models \beta$ and $x \models \gamma$ iff $x \models^p g(\beta)$ and $x \models^p g(\gamma)$ (by induction hypothesis) iff $x \models^p g(\beta) \wedge_j g(\gamma)$ iff $x \models^p g(\beta \wedge \gamma)$ (by the definition of g).
2. Case when α is $\neg\beta$: $x \models \neg\beta$ iff $x \not\models \beta$ iff $x \not\models^p g(\beta)$ (by induction hypothesis) iff $x \models^p \sim_k \sim_j g(\beta)$ iff $x \models^p g(\neg\beta)$ (by the definition of g).
3. Case when α is $\Box\beta$: $x \models \Box\beta$ iff $\forall y \in M[xRy$ implies $y \models \beta]$ iff $\forall y \in M[xRy$ implies $y \models^p g(\beta)]$ (by induction hypothesis) iff $x \models^p \Box_j g(\beta)$ iff $x \models g(\Box\beta)$ (by the definition of g).

□

Lemma 5.11. *Let g be the mapping defined in Definition 3.4. For any paraconsistent Kripke model $\langle M, R, \models^p\rangle$, we can construct a Kripke model $\langle M, R, \models\rangle$ such that for any formula α and any $x \in M$,*

$x \models^p g(\alpha)$ *iff* $x \models \alpha$.

Proof. Similar to the proof of Lemma 5.10. □

Theorem 5.12 (Semantical embedding from S4C into MML_n). *Let g be the mapping defined in Definition 3.4. For any formula α,*

α is S4C-valid iff $g(\alpha)$ is MML_n-valid.

Proof. By Lemmas 5.10 and 5.11. □

6. Duality

Definition 6.1. Let n (>1) be the positive integer determined by n-lattice, and let j be any positive integer with $j \leq n$. We fix a set Φ of propositional variables. The language $\mathcal{L}_{\mathrm{MML}}$ of MML_n is defined using Φ, $\wedge_j, \vee_j, \rightarrow_j, \leftarrow_j, \Box_j$, $\Diamond_j$ and $\sim_j$. A *dualization function* h can be defined inductively as a mapping from $\mathcal{L}_{\mathrm{MML}}$ into $\mathcal{L}_{\mathrm{MML}}$:

1. For any $p \in \Phi$, $h(p) := p$,
2. $h(\alpha \wedge_j \beta) := h(\alpha) \vee_j h(\beta)$,
3. $h(\alpha \vee_j \beta) := h(\alpha) \wedge_j h(\beta)$,
4. $h(\alpha \rightarrow_j \beta) := h(\beta) \leftarrow_j h(\alpha)$,
5. $h(\alpha \leftarrow_j \beta) := h(\beta) \rightarrow_j h(\alpha)$,
6. $h(\Box_j \alpha) := \Diamond_j h(\alpha)$,
7. $h(\Diamond_j \alpha) := \Box_j h(\alpha)$,
8. $h(\sim_j \alpha) := \sim_j h(\alpha)$.

Clauses 4 and 5 of this definition explicitly demonstrate that $\leftarrow_j$ is the *converse* dual connective to $\rightarrow_j$, and vice versa.

The following theorem represents the duality principle for MML_n:

Theorem 6.2 (Syntactical embedding from MML_ninto MML_n). *Let Γ, Δ be sets of formulas in $\mathcal{L}_{\mathrm{MML}}$, and h be the mapping defined in Definition 6.1.*

1. *$MML_n \vdash \Gamma \Rightarrow \Delta$ iff $MML_n \vdash h(\Delta) \Rightarrow h(\Gamma)$.*
2. *$MML_n -$ (cut) $\vdash \Gamma \Rightarrow \Delta$ iff $MML_n -$ (cut) $\vdash h(\Delta) \Rightarrow h(\Gamma)$.*

Proof. We show only (2) below.

• ($\Longrightarrow$): By induction on the cut-free proofs P of $\Gamma \Rightarrow \Delta$ in MML_n. We distinguish the cases according to the last inference of P, and show some cases.

1. Case ($\sim_j p \Rightarrow \sim_j p$): The last inference of P is of the form: $\sim_j p \Rightarrow \sim_j p$ for any $p \in \Phi$. In this case, we obtain $\mathrm{MML}_n -$ (cut) $\vdash h(\sim_j p) \Rightarrow h(\sim_j p)$, i.e., $\mathrm{MML}_n -$ (cut) $\vdash \sim_j p \Rightarrow \sim_j p$, by the definition of h.
2. Case ($\sim_j \wedge_j$left): The last inference of P is of the form:

$$\frac{\sim_j \alpha, \Gamma \Rightarrow \Delta \quad \sim_j \beta, \Gamma \Rightarrow \Delta}{\sim_j (\alpha \wedge_j \beta), \Gamma \Rightarrow \Delta} \ (\sim_j \wedge_j \text{left}).$$

By ind. hypothesis, we have $\mathrm{MML}_n -$ (cut) $\vdash h(\Delta) \Rightarrow h(\Gamma), h(\sim_j \alpha)$ and $\mathrm{MML}_n -$ (cut) $\vdash h(\Delta) \Rightarrow h(\Gamma), h(\sim_j \beta)$ where $h(\sim_j \alpha)$ and $h(\sim_j \beta)$ coincide with $\sim_j h(\alpha)$ and $\sim_j h(\beta)$, respectively, by the definition of h. Then, we obtain the required fact:

$$\frac{\begin{matrix}\vdots \\ h(\Delta) \Rightarrow h(\Gamma), \sim_j h(\alpha)\end{matrix} \quad \begin{matrix}\vdots \\ h(\Delta) \Rightarrow h(\Gamma), \sim_j h(\beta)\end{matrix}}{h(\Delta) \Rightarrow h(\Gamma), \sim_j (h(\alpha) \vee_j h(\beta))} \ (\sim_j \vee_j \text{right})$$

where $\sim_j (h(\alpha) \vee_j h(\beta))$ coincides with $h(\sim_j (\alpha \wedge_j \beta))$ by the definition of h.

3. Case ($\sim_j\rightarrow_j$right): The last inference of P is of the form:

$$\frac{\Gamma \Rightarrow \Delta, \sim_j\beta \quad \sim_j\alpha, \Sigma \Rightarrow \Pi}{\Gamma, \Sigma \Rightarrow \Delta, \Pi, \sim_j(\alpha\leftarrow_j\beta)}\ (\sim_j\rightarrow_j\text{right}).$$

By ind. hypothesis, we have $\mathrm{MML}_n - (\mathrm{cut}) \vdash h(\Pi) \Rightarrow h(\Sigma), h(\sim_j\alpha)$ and $\mathrm{MML}_n - (\mathrm{cut}) \vdash h(\sim_j\beta), h(\Delta) \Rightarrow h(\Gamma)$ where $h(\sim_j\alpha)$ and $h(\sim_j\beta)$ coincide with $\sim_j h(\alpha)$ and $\sim_j h(\beta)$, respectively, by the definition of h. Then, we obtain the required fact:

$$\frac{\begin{array}{c}\vdots\\ h(\Pi) \Rightarrow h(\Sigma), \sim_j h(\alpha)\end{array} \quad \begin{array}{c}\vdots\\ \sim_j h(\beta), h(\Delta) \Rightarrow h(\Gamma)\end{array}}{\sim_j(h(\beta)\leftarrow_j h(\alpha)), h(\Pi), h(\Delta) \Rightarrow h(\Sigma), h(\Gamma)}\ (\sim_j\leftarrow_j\text{left})$$

where $\sim_j(h(\beta)\leftarrow_j h(\alpha))$ coincides with $h(\sim_j(\alpha\rightarrow_j\beta))$ by the definition of h.
4. Case ($\sim_k\rightarrow_j$left): The last inference of P is of the form:

$$\frac{\Gamma \Rightarrow \Delta, \sim_k\alpha \quad \sim_k\beta, \Sigma \Rightarrow \Pi}{\sim_k(\alpha\rightarrow_j\beta), \Gamma, \Sigma \Rightarrow \Delta, \Pi}\ (\sim_k\rightarrow_j\text{left}).$$

By ind. hypothesis, we have $\mathrm{MML}_n - (\mathrm{cut}) \vdash h(\Pi) \Rightarrow h(\Sigma), h(\sim_k\beta)$ and $\mathrm{MML}_n - (\mathrm{cut}) \vdash h(\sim_k\alpha), h(\Delta) \Rightarrow h(\Gamma)$ where $h(\sim_k\alpha)$ and $h(\sim_k\beta)$ coincide with $\sim_k h(\alpha)$ and $\sim_k h(\beta)$, respectively, by the definition of h. Then, we obtain the required fact:

$$\frac{\begin{array}{c}\vdots\\ h(\Pi) \Rightarrow h(\Sigma), \sim_k h(\beta)\end{array} \quad \begin{array}{c}\vdots\\ \sim_k h(\alpha), h(\Delta) \Rightarrow h(\Gamma)\end{array}}{h(\Pi), h(\Delta) \Rightarrow h(\Sigma), h(\Gamma), \sim_k(h(\beta)\leftarrow_j h(\alpha))}\ (\sim_k\leftarrow_j\text{right})$$

where $\sim_k(h(\beta)\leftarrow_j h(\alpha))$ coincides with $h(\sim_k(\alpha\rightarrow_j\beta))$ by the definition of h.
5. Case ($\sim_k\sim_j$left): The last inference of P is of the form:

$$\frac{\Gamma \Rightarrow \Delta, \alpha}{\sim_k\sim_j\alpha, \Gamma \Rightarrow \Delta}\ (\sim_k\sim_j\text{left}).$$

By induction hypothesis, we have $\mathrm{MML}_n - (\mathrm{cut}) \vdash h(\alpha), h(\Delta) \Rightarrow h(\Gamma)$. Then, we obtain the required fact:

$$\frac{\begin{array}{c}\vdots\\ h(\alpha), h(\Delta) \Rightarrow h(\Gamma)\end{array}}{h(\Delta) \Rightarrow h(\Gamma), \sim_k\sim_j h(\alpha)}\ (\sim_k\sim_j\text{right})$$

where $\sim_k\sim_j h(\alpha)$ coincides with $h(\sim_k\sim_j\alpha)$ by the definition of h.
6. Case ($\Box_j$right): The last inference of P is of the form:

$$\frac{\Box_j\Gamma, \sim_j\Diamond_j\Sigma, \sim_k\Box_j\Pi \Rightarrow \alpha}{\Box_j\Gamma, \sim_j\Diamond_j\Sigma, \sim_k\Box_j\Pi \Rightarrow \Box_j\alpha}\ (\Box_j\text{right}).$$

By induction hypothesis, we have $\mathrm{MML}_n - (\mathrm{cut}) \vdash h(\alpha) \Rightarrow h(\Box_j\Gamma)$, $h(\sim_j\Diamond_j\Sigma)$, $h(\sim_k\Box_j\Pi)$ where $h(\Box_j\Gamma)$, $h(\sim_j\Diamond_j\Sigma)$ and $h(\sim_k\Box_j\Pi)$ coincide with $\Diamond_j h(\Gamma)$, $\sim_j\Box_j h(\Sigma)$ and $\sim_k\Diamond_j h(\Pi)$, respectively, by the definition of h. Then, we obtain the required fact:

$$\dfrac{\begin{array}{c}\vdots\\ h(\alpha) \Rightarrow \Diamond_j h(\Gamma), \sim_j\Box_j h(\Sigma), \sim_k\Diamond_j h(\Pi)\end{array}}{\Diamond_j h(\alpha) \Rightarrow \Diamond_j h(\Gamma), \sim_j\Box_j h(\Sigma), \sim_k\Diamond_j h(\Pi)}\ (\Diamond_j\text{left})$$

where $\Diamond_j h(\alpha)$ coincides with $h(\Box_j\alpha)$ by the definition of h.

7. Case ($\sim_j\Box_j$left): The last inference of P is of the form:

$$\dfrac{\sim_j\alpha \Rightarrow \Diamond_j\Gamma, \sim_j\Box_j\Sigma, \sim_k\Diamond_j\Pi}{\sim_j\Box_j\alpha \Rightarrow \Diamond_j\Gamma, \sim_j\Box_j\Sigma, \sim_k\Diamond_j\Pi}\ (\sim_j\Box_j\text{left}).$$

By induction hypothesis, we have $\mathrm{MML}_n - (\mathrm{cut}) \vdash h(\Diamond_j\Gamma), h(\sim_j\Box_j\Sigma), h(\sim_k\Diamond_j\Pi) \Rightarrow h(\sim_j\alpha)$ where $h(\Diamond_j\Gamma)$, $h(\sim_j\Box_j\Sigma)$, $h(\sim_k\Diamond_j\Pi)$ and $h(\sim_j\alpha)$ coincide with $\Box_j h(\Gamma)$, $\sim_j\Diamond_j h(\Sigma)$, $\sim_k\Box_j h(\Pi)$ and $\sim_j h(\alpha)$, respectively, by the definition of h. Then, we obtain the required fact:

$$\dfrac{\begin{array}{c}\vdots\\ \Box_j h(\Gamma), \sim_j\Diamond_j h(\Sigma), \sim_k\Box_j h(\Pi) \Rightarrow \sim_j h(\alpha)\end{array}}{\Box_j h(\Gamma), \sim_j\Diamond_j h(\Sigma), \sim_k\Box_j h(\Pi) \Rightarrow \sim_j\Diamond_j h(\alpha)}\ (\sim_j\Diamond_j\text{right})$$

where $\sim_j\Diamond_j h(\alpha)$ coincides with $h(\sim_j\Box_j\alpha)$ by the definition of h.

8. Case ($\sim_k\Box_j$right): The last inference of P is of the form:

$$\dfrac{\Box_j\Gamma, \sim_j\Diamond_j\Sigma, \sim_k\Box_j\Pi \Rightarrow \sim_k\alpha}{\Box_j\Gamma, \sim_j\Diamond_j\Sigma, \sim_k\Box_j\Pi \Rightarrow \sim_k\Box_j\alpha}\ (\sim_k\Box_j\text{right}).$$

By induction hypothesis, we have $\mathrm{MML}_n - (\mathrm{cut}) \vdash h(\sim_k\alpha) \Rightarrow h(\Box_j\Gamma), h(\sim_j\Diamond_j\Sigma), h(\sim_k\Box_j\Pi)$ where $h(\sim_k\alpha)$, $h(\Box_j\Gamma)$, $h(\sim_j\Diamond_j\Sigma)$ and $h(\sim_k\Box_j\Pi)$ coincide with $\sim_k h(\alpha)$, $\Diamond_j h(\Gamma)$, $\sim_j\Box_j h(\Sigma)$ and $\sim_k\Diamond_j h(\Pi)$, respectively, by the definition of h. Then, we obtain the required fact:

$$\dfrac{\begin{array}{c}\vdots\\ \sim_k h(\alpha) \Rightarrow \Diamond_j h(\Gamma), \sim_j\Box_j h(\Sigma), \sim_k\Diamond_j h(\Pi)\end{array}}{\sim_k\Diamond_j h(\alpha) \Rightarrow \Diamond_j h(\Gamma), \sim_j\Box_j h(\Sigma), \sim_k\Diamond_j h(\Pi)}\ (\sim_k\Diamond_j\text{left})$$

where $\sim_k\Diamond_j h(\alpha)$ coincides with $h(\sim_k\Box_j\alpha)$ by the definition of h.

• ($\Longleftarrow$): By induction on the cut-free proofs Q of $h(\Gamma) \Rightarrow h(\Delta)$ in MML_n. We distinguish the cases according to the last inference of Q, and show some cases.

1. Case ($\sim_k\sim_j$left): The last inference of Q is of the form:

$$\dfrac{h(\Gamma) \Rightarrow h(\Delta), h(\alpha)}{h(\sim_k\sim_j\alpha), h(\Gamma) \Rightarrow h(\Delta)}\ (\sim_k\sim_j\text{left})$$

where $h(\sim_k\sim_j\alpha)$ coincides with $\sim_k\sim_j h(\alpha)$ by the definition of h. By induction hypothesis, we have $\mathrm{MML}_n - (\mathrm{cut}) \vdash \alpha, \Delta \Rightarrow \Gamma$. We thus obtain the required fact:

$$\dfrac{\begin{array}{c}\vdots\\ \alpha, \Delta \Rightarrow \Gamma\end{array}}{\Delta \Rightarrow \Gamma, {\sim_k}{\sim_j}\alpha}\ ({\sim_k}{\sim_j}\text{right}).$$

2. Case $({\sim_j}{\rightarrow_j}\text{right})$: The last inference of Q is of the form:

$$\dfrac{h(\Gamma) \Rightarrow h(\Delta), h({\sim_j}\beta) \quad h({\sim_j}\alpha), h(\Sigma) \Rightarrow h(\Pi)}{h(\Gamma), h(\Sigma) \Rightarrow h(\Delta), h(\Pi), h({\sim_j}(\beta{\leftarrow_j}\alpha))}\ ({\sim_j}{\rightarrow_j}\text{right})$$

where $h({\sim_j}\alpha)$, $h({\sim_j}\beta)$ and $h({\sim_j}(\beta{\leftarrow_j}\alpha))$ coincide with ${\sim_j}h(\alpha)$, ${\sim_j}h(\beta)$ and ${\sim_j}(h(\alpha){\rightarrow_j}h(\beta))$, respectively, by the definition of h. By induction hypothesis, we have , $\mathrm{MML}_n - (\text{cut}) \vdash \Pi \Rightarrow \Sigma, {\sim_j}\alpha$ and $\mathrm{MML}_n - (\text{cut}) \vdash {\sim_j}\beta, \Delta \Rightarrow \Gamma$. We thus obtain the required fact:

$$\dfrac{\begin{array}{c}\vdots\\ \Pi \Rightarrow \Sigma, {\sim_j}\alpha\end{array} \quad \begin{array}{c}\vdots\\ {\sim_j}\beta, \Delta \Rightarrow \Gamma\end{array}}{{\sim_j}(\beta{\leftarrow_j}\alpha), \Pi, \Delta \Rightarrow \Sigma, \Gamma}\ ({\sim_j}{\leftarrow_j}\text{left}).$$

3. Case $({\sim_k}{\rightarrow_j}\text{left})$: The last inference of Q is of the form:

$$\dfrac{h(\Gamma) \Rightarrow h(\Delta), h({\sim_k}\alpha) \quad h({\sim_k}\beta), h(\Sigma) \Rightarrow h(\Pi)}{h({\sim_k}(\beta{\leftarrow_j}\alpha)), h(\Gamma), h(\Sigma) \Rightarrow h(\Delta), h(\Pi)}\ ({\sim_k}{\rightarrow_j}\text{left})$$

where $h({\sim_k}\alpha)$, $h({\sim_k}\beta)$ and $h({\sim_k}(\beta{\leftarrow_j}\alpha))$ coincide with ${\sim_k}h(\alpha)$, ${\sim_k}h(\beta)$ and ${\sim_k}(h(\alpha){\rightarrow_j}h(\beta))$, respectively, by the definition of h. By induction hypothesis, we have , $\mathrm{MML}_n - (\text{cut}) \vdash \Pi \Rightarrow \Sigma, {\sim_k}\beta$ and $\mathrm{MML}_n - (\text{cut}) \vdash {\sim_k}\alpha, \Delta \Rightarrow \Gamma$. We thus obtain the required fact:

$$\dfrac{\begin{array}{c}\vdots\\ \Pi \Rightarrow \Sigma, {\sim_k}\beta\end{array} \quad \begin{array}{c}\vdots\\ {\sim_k}\alpha, \Delta \Rightarrow \Gamma\end{array}}{\Pi, \Delta \Rightarrow \Sigma, \Gamma, {\sim_k}(\beta{\leftarrow_j}\alpha)}\ ({\sim_k}{\leftarrow_j}\text{right}).$$

4. Case $({\sim_j}\Box_j\text{left})$: The last inference of Q is of the form:

$$\dfrac{h({\sim_j}\alpha) \Rightarrow h(\Box_j\Gamma), h({\sim_j}\Diamond_j\Sigma), h({\sim_k}\Box_j\Pi)}{h({\sim_j}\Diamond_j\alpha) \Rightarrow h(\Box_j\Gamma), h({\sim_j}\Diamond_j\Sigma), h({\sim_k}\Box_j\Pi)}\ ({\sim_j}\Box_j\text{left})$$

where $h({\sim_j}\Diamond_j\alpha)$, $h(\Box_j\Gamma)$, $h({\sim_j}\Diamond_j\Sigma)$ and $h({\sim_k}\Box_j\Pi)$ coincides with ${\sim_j}\Box_j h(\alpha)$, $\Diamond_j h(\Gamma)$, ${\sim_j}\Box_j h(\Sigma)$ and ${\sim_k}\Diamond_j h(\Pi)$, respectively, by the definition of h. By induction hypothesis, we have $\mathrm{MML}_n - (\text{cut}) \vdash \Box_j\Gamma, {\sim_j}\Diamond_j\Sigma, {\sim_k}\Box_j\Pi \Rightarrow {\sim_j}\alpha$. We thus obtain the required fact:

$$\dfrac{\begin{array}{c}\vdots\\ \Box_j\Gamma, {\sim_j}\Diamond_j\Sigma, {\sim_k}\Box_j\Pi \Rightarrow {\sim_j}\alpha\end{array}}{\Box_j\Gamma, {\sim_j}\Diamond_j\Sigma, {\sim_k}\Box_j\Pi \Rightarrow {\sim_j}\Diamond_j\alpha}\ ({\sim_j}\Diamond_j\text{right}).$$

$\Box$

7. Concluding Remarks

The very idea of combining many-valued and modal logics within a joint framework is not new. Some traditional results in that respect are found, for example, in [8,9] and the references therein. Nevertheless, the modal extensions of the bilattice, trilattice and tetralattice logics have not yet been studied intensively. Some many-valued modal logics over finite residuated lattices were considered by Bou et al. [5], with a special attention to some basic classes of Kripke frames and their axiomatizations. One may refer also to [24,25,28], where some modal extensions of Belnap–Dunn four-valued logic are considered, and certain properties of such logics from a proof-theoretic, semantic and algebraic standpoints are analyzed.

In particular, Odintsov and Wansing [25] introduce some four- and three-valued modal logics, which are extensions of Belnap–Dunn four-valued logic and its three-valued variant, providing them with the sound and complete tableau calculi, Kripke semantics and modal algebras with twist structures. A family of four-valued modal logics, which are modal extensions of Belnap–Dunn four-valued logic, has recently been studied by Rivieccio et al. [28], by considering the many-valued Kripke structures and their counterpart modal algebras in the sense of the topological duality theory. Odintsov and Speranski [24] deal with a Belnapian version BK of the least normal modal logic K with addition of strong negation, and carry out a systematic study of the lattices of logics containing BK.

By way of comparison, it can be said that each of the aforementioned works and our present study are based on different algebraic structures: residuated lattices are adopted in [5], bilattices or Belnap lattices are adopted in [25], (modal) bilattices are adopted in [28], BK-lattices with twist structure were adopted in [24], and multilattices are adopted in our study. From the point of view of Kripke semantics, both many-valued accessibility relations and many-valued valuations have been introduced and studied. For example, many-valued accessibility relations were introduced by Fitting [8,9]. Our Kripke semantics uses many-valued valuations, since these are simple and compatible with the standard Kripke semantics for normal modal logics.

In the present paper we have proposed also a more systematic approach to modalities in a context of multilattice logics, without limitation to the dimension of a multilattice under consideration. A distinctive feature of this approach is an admission of exactly n modalities of certain kind, each corresponding to an ordering relation in a given multilattice.

The resulting Gentzen-type sequent system MML_n presents a simple and uniform framework for dealing with modal notions in a generalized paraconsistent perspective. We equipped MML_n with modalities of S4-type, but it could also be interesting to examine other types of modalities incorporated in a multilattice framework.

For example, it is possible to obtain from MML_n the modalities of S5-type by abandoning the single formula restriction in the sequents of the rules

($\Box_j$right), ($\sim_j\Diamond_j$right), ($\sim_k\Box_j$right), ($\Diamond_j$left), ($\sim_j\Box_j$left), ($\sim_k\Diamond_j$left), and supplementing them with an appropriate context. Thus, the rule ($\Box_j$right) would take then the following form:

$$\frac{\Box_j\Gamma, \sim_j\Diamond_j\Sigma, \sim_k\Box_j\Pi \Rightarrow \Box_j\Delta, \sim_j\Diamond_j\Theta, \sim_k\Box_j\Lambda, \alpha}{\Box_j\Gamma, \sim_j\Diamond_j\Sigma, \sim_k\Box_j\Pi \Rightarrow \Box_j\Delta, \sim_j\Diamond_j\Theta, \sim_k\Box_j\Lambda, \Box_j\alpha} \ ,$$

and similarly for other rules just mentioned.

The embeddings of this modified system with S5-modalities into S5C (the S5 analogue of S4C) and vice versa, as well as almost all the results demonstrated in the present paper with respect to MML_n, should equally hold for the modified systems with S5-modalities. It is known, however, that the Gentzen-type sequent calculus for S5, obtained from S4 by abandoning the above mentioned restriction, is not cut-free (see e.g., [27]), and hence, cut-elimination fails in the modified system as well. Finding thus a natural formulation of a cut-free modal multilattice logic with S5-modalities is an interesting task deserving special investigation.

References

[1] Almukdad, A., Nelson, D.: Constructible falsity and inexact predicates. J. Symb. Log. **49**(1), 231–233 (1984)

[2] Arieli, O., Avron, A.: Reasoning with logical bilattices. J. Log. Lang. Inf. **5**, 25–63 (1996)

[3] Belnap, N.: A useful four-valued logic. In: Epstein, G., Dunn, J.M. (eds.) Modern Uses of Multiple-Valued Logic, pp. 5–37. Reidel, Dordrecht (1977)

[4] Belnap, N.: How a computer should think. In: Ryle, G. (ed.) Contemporary Aspects of Philosophy, pp. 30–56. Oriel Press, Stocksfield (1977)

[5] Bou, F., Esteva, F., Godo, L., Rodríguez, R.: On the minimum many-valued modal logic over a finite residuated lattice. J. Log. Comput. **21**, 739–790 (2011)

[6] Cattaneo, G., Ciucci, D.: Lattices with interior and closure operators and abstract approximation spaces. In: Peters, J.F. et al. (eds.) Transactions on Rough Sets X, LNCS vol. 5656, pp. 67–116 (2009)

[7] Dunn, J.M.: Intuitive semantics for first-degree entailment and 'coupled trees'. Philos. Stud. **29**, 149–168 (1976)

[8] Fitting, M.: Many-valued modal logics. Fundam. Inf. **15**, 235–254 (1991)

[9] Fitting, M.: Many-valued modal logics II. Fundam. Inf. **17**, 55–73 (1992)

[10] Ginsberg, M.: Multi-valued logics. In: Proceedings of AAAI-86, Fifth National Conference on Artificial Intellegence. Morgan Kaufman Publishers, Los Altos, pp. 243–247 (1986)

[11] Ginsberg, M.: Multivalued logics: a uniform approach to reasoning in AI. Comput. Intell. **4**, 256–316 (1988)

[12] Gurevich, Y.: Intuitionistic logic with strong negation. Stud. Log. **36**, 49–59 (1977)

[13] Kamide, N.: Proof systems combining classical and paraconsistent negations. Stud. Log. **91**(2), 217–238 (2009)

[14] Kamide, N.: Embedding-based methods for trilattice logic. In: Proceedings of the 34rd IEEE International Symposium on Multiple-Valued Logic (ISMVL 2013), pp. 237–242 (2013)

[15] Kamide, N.: A hierarchy of weak double negations. Stud. Log. **101**(6), 1277–1297 (2013)

[16] Kamide, N.: Trilattice logic: an embedding-based approach. J. Log. Comput. **25**(3), 581–611 (2015)

[17] Kamide, N., Shramko, Y.: Embedding from multilattice logic into classical logic and vice versa. J. Logic Comput. **27**(5), 1549–1575 (2017). doi:10.1093/logcom/exw015

[18] Kamide, N., Wansing, H.: Symmetric and dual paraconsistent logics. Log. Log. Philos. **19**(1–2), 7–30 (2010)

[19] Kamide, N., Shramko, Y., Wansing, H.: Kripke completeness of bi-intuitionistic multilattice logic and its connexive variant. Stud. Log. (to appear)

[20] Lemmon, E.J.: Algebraic semantics for modal logics I. J. Symb. Log. **31**, 46–65 (1966)

[21] McKinsey, J.C.C., Tarski, A.: The algebra of topology. Ann. Math. **45**, 141–191 (1944)

[22] McKinsey, J.C.C., Tarski, A.: Some theorems about the sentential calculi of Lewis and Heyting. J. Symb. Log. **13**, 1–15 (1948)

[23] Nelson, D.: Constructible falsity. J. Symb. Log. **14**, 16–26 (1949)

[24] Odintsov, S.P., Speranski, S.: The lattice of Belnapian modal logics: special extensions and counterparts. Log. Log. Philos. **25**, 3–33 (2016)

[25] Odintsov, S.P., Wansing, H.: Modal logics with Belnapian truth values. J. Appl. Non-Class. Log. **20**, 279–301 (2010)

[26] Ohnishi, M., Matsumoto, K.: Gentzen method in modal calculi. Osaka Math. J. **9**, 113–130 (1957)

[27] Ohnishi, M., Matsumoto, K.: Gentzen method in modal calculi II. Osaka Math. J. **11**, 115–120 (1959)

[28] Rivieccio, U., Jung, A., Jansana, R.: Four-valued modal logic: Kripke semantics and duality. J. Log. Comput. **27**, 155–199 (2017)

[29] Rautenberg, W.: Klassische und nicht-klassische Aussagenlogik. Vieweg, Braunschweig (1979)

[30] Shramko, Y.: Truth, falsehood, information and beyond: the American plan generalized. In: Bimbo, K. (ed.) J. Michael Dunn on Information Based Logics, Outstanding Contributions to Logic. Springer, Dordrecht, pp. 191–212 (2016)

[31] Shramko, Y.: A modal translation for dual-intuitionistic logic. Rev. Symb. Log. **9**, 251–265 (2016)

[32] Shramko, Y., Dunn, J.M., Takenaka, T.: The trilattice of constructive truth values. J. Log. Comput. **11**(6), 761–788 (2001)

[33] Shramko, Y., Wansing, H.: Some useful sixteen-valued logics: how a computer network should think. J. Philos. Log. **34**, 121–153 (2005)

[34] Shramko, Y., Wansing, H.: Truth and Falsehood. An Inquiry into Generalized Logical Values. Springer, Dordrecht (2011)

[35] Schroeder-Heister, P.: Schluß und Umkehrschluß: ein Beitrag zur Definitionstheorie. In.: Gethmann, C.F. (ed.) Deutsches Jahrbuch Philosophie 02. Lebenswelt und Wissenschaft. Felix Meiner Verlag, Hamburg, pp. 1065–1092 (2009)

[36] Vorob'ev, N.N: A constructive propositional calculus with strong negation (in Russian). Doklady Akademii Nauk SSSR **85**, pp. 465–468 (1952)

[37] Wansing, H.: Constructive negation, implication, and co-implication. J. Appl. Non-Class. Log. **18**(2–3), 341–364 (2008)

[38] Zaitsev, D.: A few more useful 8-valued logics for reasoning with tetralattice EIGHT4. Stud. Log. **92**(2), 265–280 (2009)

Norihiro Kamide
Department of Information and Electronic Engineering
Teikyo University
Toyosatodai 1-1
Utsunomiya
Tochigi 320-8551
Japan
e-mail: drnkamide08@kpd.biglobe.ne.jp

Yaroslav Shramko
Department of Philosophy
Kryvyi Rih State Pedagogical University
Kryvyi Rih 50086
Ukraine
e-mail: shramko@rocketmail.com

Received: October 27, 2016.
Accepted: May 25, 2017.