Norihiro Kamide
Yaroslav Shramko
Heinrich Wansing

Kripke Completeness of Bi-intuitionistic Multilattice Logic and its Connexive Variant

Abstract. In this paper, *bi-intuitionistic multilattice logic*, which is a combination of multilattice logic and the bi-intuitionistic logic also known as Heyting–Brouwer logic, is introduced as a Gentzen-type sequent calculus. A Kripke semantics is developed for this logic, and the completeness theorem with respect to this semantics is proved via theorems for embedding this logic into bi-intuitionistic logic. The logic proposed is an extension of *first-degree entailment logic* and can be regarded as a bi-intuitionistic variant of the original classical multilattice logic determined by the algebraic structure of multilattices. Similar completeness and embedding results are also shown for another logic called *bi-intuitionistic connexive multilattice logic*, obtained by replacing the connectives of intuitionistic implication and co-implication with their connexive variants.

Keywords: First-degree entailment logic, Multilattices, Bi-intuitionistic logic, Connexive logic.

Introduction

The aim of this paper is to expand the realm of the *first-degree entailment logic* (FDE) first presented by Anderson and Belnap [2,3,5], and justified semantically by Dunn [9] and Belnap [7,8]. FDE, also called *Belnap and Dunn's four-valued logic*, is widely considered to be very useful for a number of different purposes in philosophy and computer science. This logic is based on a compound algebraic structure known as a *bilattice* [10–12] and allows various natural generalizations that can be pursued along different lines.

One such line leads to the notion of a *trilattice* [32] and—more generally—*multilattice*, resulting in logical systems comprising more than one consequence relation, notably bi-consequence systems [25,26,33] and multi-consequence systems [31]. Another way of generalizing FDE was developed by Arieli and Avron [6] under the name of *logical bilattices*, which determine

Special Issue: 40 years of FDE
Edited by **Hitoshi Omori and Heinrich Wansing**

Studia Logica (2017) 105: 1193–1219
DOI: 10.1007/s11225-017-9752-x

logical systems handling simultaneously operations under *both* bilattice orderings. A transition to *logical multilattices* seems then very natural as well, and one can find in [31] a sketch of a theory of *logical multilattices* based on lattices with n ($\in \mathbb{N}$) ordering relations.

Exactly as logical systems resting on the algebraic structure of bilattices can be collectively referred to as *bilattice logics*, the more general realm of *multilattice logics* can be considered as consisting of various systems induced by multilattices and their ordering relations. These relations can represent various possible characterizations of a given set of truth values in terms of information, truth, falsity, constructivity, modality, certainty, etc. In [17], some theorems for syntactic and semantic embeddings of a Gentzen-type sequent calculus ML_n for multilattice logic into a Gentzen-type sequent calculus LK for classical logic and vice versa were shown, and the cut-elimination, decidability and completeness theorems for ML_n were proved via these embedding theorems.

In the present paper, we develop a bi-intuitionistic version of classical multilattice logic, which comprises both intuitionistic-type connectives and their duals instead of the classical ones. An intuitionistic formalization is known to be appropriate for representing consistent information based on Kripke semantics, whereas the dual-intuitionistic co-negation falls into the category of paraconsistent operators. In this respect, we need both types of connectives—most notably, intuitionistic implication and co-implication—in the language of the underlying multilattice logic. We thus adopt bi-intuitionistic logic as the underlying logic for the multilattice logical formalism developed in this paper. We show also how another logic, called *bi-intuitionistic connexive multilattice logic*, can be obtained by replacing the connectives of intuitionistic implication and co-implication with their connexive variants. A Kripke semantics can be developed for all these logics, and the completeness theorem with respect to this semantics can be proved through the embedding technique. The results of this paper include a combined extension of the results of both the paper [17] for classical multilattice logic and the paper [20] for bi-intuitionistic connexive logic.

Bi-intuitionistic logic, also called *Heyting–Brouwer logic*, was originally introduced by Rauszer [27–29], who proved the corresponding Kripke completeness theorem. Various sequent calculi for bi-intuitionistic logic have been proposed by several researchers (see, e.g., [13]), since the original Gentzen-type sequent calculus by Rauszer does not enjoy cut-elimination. Bi-intuitionistic logic is also known to be a logic that has a translation into the future-past tense logic KtT4 [21,43]. A modal logic based on bi-intuitionistic logic was also studied by Łukowski in [22]. An alternative bi-intuitionistic logic with a different understanding of co-implication, called 2Int, was proposed in [39, 41] for combining the notions of verification and its dual. A connexive variant of 2Int is studied in [42].

Connexive logics are known to be nonclassical logics motivated by the idea of formalizing a coherence between the premises and conclusions of valid inferences (as in relevance logic) or between formulas of a certain form; for a survey see [40]. Some modern perspectives of connexive logics were given by Angell [4] and McCall [23], although the origins of connexive logics ascend to Aristotle and Boethius. An *intuitionistic (or constructive) connexive modal logic*, which is a constructive connexive analogue of the smallest normal modal logic K, was introduced in [37]. Bi-intuitionistic connexive logic, which is an integration of bi-intuitionistic logic and the non-modal fragment of the intuitionistic connexive modal logic, was originally introduced in [38] as a cut-free display calculus. A version of bi-intuitionistic connexive logic, BCL, also called *connexive Heyting–Brouwer logic*, was introduced in [20] as a Gentzen-type sequent calculus, and the completeness theorem with respect to a Kripke semantics was proved for BCL.

The subsequent exposition can be summarized as follows. In Section 1 we recapitulate the main idea of logical multilattices and the notion of a multilattice logic. In Section 2, Gentzen-type sequent calculi BML_n and BL for bi-intuitionistic multilattice (n-lattice) logic and bi-intuitionistic logic, respectively, are introduced, and Kripke semantics for these logics are introduced and explained. In Section 3, theorems for a syntactic and semantic embedding of BML_n into BL and vice versa are proved, and the Kripke completeness theorem for BML_n is obtained from these embedding theorems. In Section 4, similar completeness and embedding results are shown for bi-intuitionistic connexive multilattice logic.

1. The Idea of Logical Multilattices

Let us first recall the definition of a multilattice from [31] as an algebraic structure with several partial orderings:

Definition 1.1. An n-dimensional **multilattice** (or just n-lattice) is a structure

$$\mathcal{M}_n = (S, \sqsubseteq_1, \ldots, \sqsubseteq_n),$$

where S is a non-empty set, and $\sqsubseteq_1, \ldots, \sqsubseteq_n$ are partial orderings each giving S the structure of a lattice, determining thus for each of the n lattices the corresponding pairs of meet and join operations denoted by $\langle \sqcap_1, \sqcup_1 \rangle$, ..., $\langle \sqcap_n, \sqcup_n \rangle$.

Besides meets and joins, a multilattice may come with inversion operations defined with respect to each ordering relation:

DEFINITION 1.2. Let $\mathcal{M}_n = (S, \sqsubseteq_1, \dots, \sqsubseteq_n)$ be a multilattice. Then for any $j \leq n$ an unary operation $-_j$ on S is said to be a (pure) j-inversion iff for any $k \leq n$, $k \neq j$ the following conditions are satisfied:

$$\begin{array}{ll} (anti) & x \sqsubseteq_j y \Rightarrow -_j y \sqsubseteq_j -_j x; \\ (iso) & x \sqsubseteq_k y \Rightarrow -_j x \sqsubseteq_k -_j y; \\ (per2) & -_j -_j x = x. \end{array}$$

Thus, in a multilattice-framework any inversion defined relative to some partial order is an involution operation, antitone with respect to this particular order and isotone with respect to all the remaining orderings. The following notion of a multifilter is a generalization of the notion of a bifilter from [6]:

DEFINITION 1.3. Let $\mathcal{M}_n = (S, \sqsubseteq_1, \dots, \sqsubseteq_n)$ be an n-lattice, with pairs of meet and join operations $\langle \sqcap_1, \sqcup_1 \rangle$, ..., $\langle \sqcap_n, \sqcup_n \rangle$. An **n-filter** (multifilter) on $\mathcal{M}_n$ is a nonempty proper subset $\mathcal{F}_n \subset S$, such that for every $j \leq n$:

$$x \sqcap_j y \in \mathcal{F}_n \Leftrightarrow x \in \mathcal{F}_n \text{ and } y \in \mathcal{F}_n.$$

A multifilter $\mathcal{F}_n$ is said to be **prime** iff it satisfies for every $j \leq n$:

$$x \sqcup_j y \in \mathcal{F}_n \Leftrightarrow x \in \mathcal{F}_n \text{ or } y \in \mathcal{F}_n.$$

A pair $(\mathcal{M}_n, \mathcal{F}_n)$ is called a **logical n-lattice** (logical multilattice) iff $\mathcal{M}_n$ is a multilattice, and $\mathcal{F}_n$ is a prime multifilter on $\mathcal{M}_n$.

If we are interested in n-lattices with inversions existing for every $j \leq n$, the stronger notions of an *ultramultifilter* and *ultralogical multilattice* are in order.

DEFINITION 1.4. Let $\mathcal{M}_n = (S, \sqsubseteq_1, \dots, \sqsubseteq_n)$ be an n-lattice, with j-inversions defined with respect to every $\sqsubseteq_j$ ($j \leq n$). Then $\mathcal{F}_n$ is an **n-ultrafilter** (ultramultifilter) on $\mathcal{M}_n$ if and only if it is a prime multifilter, such that for every $j, k \leq n$, $j \neq k$: $x \in \mathcal{F}_n \Leftrightarrow -_j -_k x \notin \mathcal{F}_n$. A pair $(\mathcal{M}_n, \mathcal{F}_n)$ is called an **ultralogical n-lattice** (ultralogical multilattice) iff $\mathcal{M}_n$ is a multilattice, and $\mathcal{F}_n$ is an ultramultifilter on $\mathcal{M}_n$.

Intuitively, meet, join and inversion with respect to an ordering in a given multilattice determine the corresponding connectives of conjunction, disjunction and negation. Thus, an n-lattice with inversions generates exactly n basic pairs of conjunctions and disjunctions, accompanied with at least n negation-like operators. The respective (ultra)multifilter represents then the set of designated elements used for defining the entailment relation, as is standardly done in many-valued logic.[1] Having an ultralogical multilattice, the corresponding *minimal multilattice logic* can be conceived as a system which operates solely with the connectives of conjunctions, disjunctions and negations. For any n (>1), such multilattice logic was formalized in [31] as a Gentzen-style sequent system GML_n, and some of the features of the systems GML_n were investigated further in [17]. Moreover, in [17] this minimal multilattice logic was extended by quantifiers, implications and co-implications, determined for each $j \leq n$ by their classical-type inference rules. These additional logical constants receive then not a lattice-theoretic, but a model-semantic characterization by suitable model constructions.

In the next section we will explore a way of extending the basic set of multilattice-operations by the connectives of intuitionistic and dual-intuitionistic implications to obtain the desired bi-intuitionistic multilattice logic. It also turns out that the multilattice inversion operations can be then effectively used for modeling intuitionistic and dual-intuitionistic negations. As a result, we obtain multi-consequence constructive and connexive extensions of the basic paraconsistent logic FDE.

2. Sequent Calculus and Kripke Semantics for Bi-intuitionistic Multilattice Logic

Formulas of *bi-intuitionistic n-lattice logic* are constructed from countably many propositional variables by logical connectives $\wedge_j$, $\vee_j$, $\rightarrow_j$, $\leftarrow_j$ and $\sim_j$ for every $j \leq n$, where n is the positive integer determined by a given n-lattice. Having some $j \leq n$, we can speak of these connectives as of j-conjunction, j-disjunction, j-implication, j-co-implication and j-negation correspondingly. Throughout this paper in any concatenation of two connectives $\sim_j\sim_k$ or $\sim_k\sim_j$, where $j \neq k$, it is presupposed that $j < k$. In what follows, we use small letters $p, q, \ldots$ to denote propositional variables, Greek small letters $\alpha, \beta, \ldots$ to denote formulas, and Greek capital letters $\Gamma, \Delta, \ldots$ to denote finite (possibly empty) sets of formulas. We use the symbol $\equiv$ to

[1] In [33,34], entailment is defined as order-preservation under all valuations of atomic formulas. In the case of FDE, preservation of the truth order in the smallest non-trivial bilattice $\mathcal{FOUR}$ and preservation of the designated values "told true only" and "told both true and false" from that bilattice coincide.

represent the equality of symbols. A *sequent* is an expression of the form $\Gamma \Rightarrow \Delta$. An expression $\alpha \Leftrightarrow \beta$ is used to represent the sequents $\alpha \Rightarrow \beta$ and $\beta \Rightarrow \alpha$. An expression $L \vdash \Gamma \Rightarrow \Delta$ means that the sequent $\Gamma \Rightarrow \Delta$ is provable in a sequent calculus L.

The Gentzen-type sequent calculus BML_n for bi-intuitionistic n-lattice logic is defined as follows.

DEFINITION 2.1 (BML_n). Let n (>1) be the positive integer determined by some n-lattice, j, k be any positive integers with $j, k \leq n$ and $j \neq k$ (and $j < k$ in $\sim_j\sim_k$ and $\sim_k\sim_j$).

The initial sequents of BML_n are of the following form, for any propositional variable p,

$$p \Rightarrow p \qquad\qquad \sim_j p \Rightarrow \sim_j p.$$

The structural inference rules of BML_n are of the form:

$$\frac{\Gamma \Rightarrow \Delta, \alpha \quad \alpha, \Sigma \Rightarrow \Pi}{\Gamma, \Sigma \Rightarrow \Delta, \Pi} \text{ (cut)}$$

$$\frac{\Gamma \Rightarrow \Delta}{\alpha, \Gamma \Rightarrow \Delta} \text{ (we-left)} \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \alpha} \text{ (we-right)}.$$

The logical inference rules of BML_n are of the form:

$$\frac{\Gamma \Rightarrow \Delta, \alpha \quad \beta, \Sigma \Rightarrow \Pi}{\alpha{\rightarrow_j}\beta, \Gamma, \Sigma \Rightarrow \Delta, \Pi} \text{ (}{\rightarrow_j}\text{left)} \qquad \frac{\alpha, \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha{\rightarrow_j}\beta} \text{ (}{\rightarrow_j}\text{right)}$$

$$\frac{\alpha \Rightarrow \Delta, \beta}{\alpha{\leftarrow_j}\beta \Rightarrow \Delta} \text{ (}{\leftarrow_j}\text{left)} \qquad \frac{\Gamma \Rightarrow \Delta, \alpha \quad \beta, \Sigma \Rightarrow \Pi}{\Gamma, \Sigma \Rightarrow \Delta, \Pi, \alpha{\leftarrow_j}\beta} \text{ (}{\leftarrow_j}\text{right)}$$

$$\frac{\alpha, \beta, \Gamma \Rightarrow \Delta}{\alpha\wedge_j\beta, \Gamma \Rightarrow \Delta} \text{ (}\wedge_j\text{left)} \qquad \frac{\Gamma \Rightarrow \Delta, \alpha \quad \Gamma \Rightarrow \Delta, \beta}{\Gamma \Rightarrow \Delta, \alpha\wedge_j\beta} \text{ (}\wedge_j\text{right)}$$

$$\frac{\alpha, \Gamma \Rightarrow \Delta \quad \beta, \Gamma \Rightarrow \Delta}{\alpha\vee_j\beta, \Gamma \Rightarrow \Delta} \text{ (}\vee_j\text{left)} \qquad \frac{\Gamma \Rightarrow \Delta, \alpha, \beta}{\Gamma \Rightarrow \Delta, \alpha\vee_j\beta} \text{ (}\vee_j\text{right)}$$

$$\frac{\sim_j\beta \Rightarrow \Delta, \sim_j\alpha}{\sim_j(\alpha{\rightarrow_j}\beta) \Rightarrow \Delta} \text{ (}{\sim_j}{\rightarrow_j}\text{left)} \qquad \frac{\Gamma \Rightarrow \Delta, \sim_j\beta \quad \sim_j\alpha, \Sigma \Rightarrow \Pi}{\Gamma, \Sigma \Rightarrow \Delta, \Pi, \sim_j(\alpha{\rightarrow_j}\beta)} \text{ (}{\sim_j}{\rightarrow_j}\text{right)}$$

$$\frac{\Gamma \Rightarrow \Delta, \sim_j\beta \quad \sim_j\alpha, \Sigma \Rightarrow \Pi}{\sim_j(\alpha{\leftarrow_j}\beta), \Gamma, \Sigma \Rightarrow \Delta, \Pi} \text{ (}{\sim_j}{\leftarrow_j}\text{left)} \qquad \frac{\sim_j\beta, \Gamma \Rightarrow \sim_j\alpha}{\Gamma \Rightarrow \sim_j(\alpha{\leftarrow_j}\beta)} \text{ (}{\sim_j}{\leftarrow_j}\text{right)}$$

$$\frac{\sim_j\alpha, \Gamma \Rightarrow \Delta \quad \sim_j\beta, \Gamma \Rightarrow \Delta}{\sim_j(\alpha\wedge_j\beta), \Gamma \Rightarrow \Delta} \text{ (}{\sim_j}\wedge_j\text{left)} \qquad \frac{\Gamma \Rightarrow \Delta, \sim_j\alpha, \sim_j\beta}{\Gamma \Rightarrow \Delta, \sim_j(\alpha\wedge_j\beta)} \text{ (}{\sim_j}\wedge_j\text{right)}$$

$$\frac{\sim_j\alpha, \sim_j\beta, \Gamma \Rightarrow \Delta}{\sim_j(\alpha\vee_j\beta), \Gamma \Rightarrow \Delta}\ (\sim_j\vee_j\text{left}) \qquad \frac{\Gamma \Rightarrow \Delta, \sim_j\alpha \quad \Gamma \Rightarrow \Delta, \sim_j\beta}{\Gamma \Rightarrow \Delta, \sim_j(\alpha\vee_j\beta)}\ (\sim_j\vee_j\text{right})$$

$$\frac{\alpha, \Gamma \Rightarrow \Delta}{\sim_j\sim_j\alpha, \Gamma \Rightarrow \Delta}\ (\sim_j\sim_j\text{left}) \qquad \frac{\Gamma \Rightarrow \Delta, \alpha}{\Gamma \Rightarrow \Delta, \sim_j\sim_j\alpha}\ (\sim_j\sim_j\text{right})$$

$$\frac{\Gamma \Rightarrow \Delta, \sim_k\alpha \quad \sim_k\beta, \Sigma \Rightarrow \Pi}{\sim_k(\alpha\rightarrow_j\beta), \Gamma, \Sigma \Rightarrow \Delta, \Pi}\ (\sim_k\rightarrow_j\text{left}) \qquad \frac{\sim_k\alpha, \Gamma \Rightarrow \sim_k\beta}{\Gamma \Rightarrow \sim_k(\alpha\rightarrow_j\beta)}\ (\sim_k\rightarrow_j\text{right})$$

$$\frac{\sim_k\alpha \Rightarrow \Delta, \sim_k\beta}{\sim_k(\alpha\leftarrow_j\beta) \Rightarrow \Delta}\ (\sim_k\leftarrow_j\text{left}) \qquad \frac{\Gamma \Rightarrow \Delta, \sim_k\alpha \quad \sim_k\beta, \Sigma \Rightarrow \Pi}{\Gamma, \Sigma \Rightarrow \Delta, \Pi, \sim_k(\alpha\leftarrow_j\beta)}\ (\sim_k\leftarrow_j\text{right})$$

$$\frac{\sim_k\alpha, \sim_k\beta, \Gamma \Rightarrow \Delta}{\sim_k(\alpha\wedge_j\beta), \Gamma \Rightarrow \Delta}\ (\sim_k\wedge_j\text{left}) \qquad \frac{\Gamma \Rightarrow \Delta, \sim_k\alpha \quad \Gamma \Rightarrow \Delta, \sim_k\beta}{\Gamma \Rightarrow \Delta, \sim_k(\alpha\wedge_j\beta)}\ (\sim_k\wedge_j\text{right})$$

$$\frac{\sim_k\alpha, \Gamma \Rightarrow \Delta \quad \sim_k\beta, \Gamma \Rightarrow \Delta}{\sim_k(\alpha\vee_j\beta), \Gamma \Rightarrow \Delta}\ (\sim_k\vee_j\text{left}) \qquad \frac{\Gamma \Rightarrow \Delta, \sim_k\alpha, \sim_k\beta}{\Gamma \Rightarrow \Delta, \sim_k(\alpha\vee_j\beta)}\ (\sim_k\vee_j\text{right})$$

$$\frac{\Gamma \Rightarrow \alpha}{\sim_j\sim_k\alpha, \Gamma \Rightarrow}\ (\sim_j\sim_k\text{left}) \qquad \frac{\alpha, \Gamma \Rightarrow}{\Gamma \Rightarrow \sim_j\sim_k\alpha}\ (\sim_j\sim_k\text{right})$$

$$\frac{\Rightarrow \Gamma, \alpha}{\sim_k\sim_j\alpha \Rightarrow \Gamma}\ (\sim_k\sim_j\text{left}) \qquad \frac{\alpha \Rightarrow \Gamma}{\Rightarrow \Gamma, \sim_k\sim_j\alpha}\ (\sim_k\sim_j\text{right}).$$

Some remarks on BML_n may be instrumental.

1. ($\rightarrow_j$right) and ($\leftarrow_j$left) in BML_n have the single-succedent condition and the single-antecedent condition, respectively. These rules are the inference rules of the standard Gentzen-type sequent calculi for intuitionistic and dual-intuitionistic logics. The same conditions are also imposed on ($\sim_j\rightarrow_j$left), ($\sim_j\leftarrow_j$right), ($\sim_k\rightarrow_j$right), ($\sim_k\leftarrow_j$left), as well as ($\sim_j\sim_k$left), ($\sim_j\sim_k$right), ($\sim_k\sim_j$left) and ($\sim_k\sim_j$right).
2. BML_n can be regarded as a bi-intuitionistic modification of the original system GML_n introduced in [31] and of its slightly modified version ML_n introduced in [17]. Namely, BML_n is obtained from ML_n by modifying inference rules concerning $\sim_k\sim_j$, and adding rules concerning $\rightarrow_j$ and $\leftarrow_j$.
3. BML_n is also a modified bi-intuitionistic version of the propositional fragment of the classical first-order logic FML_n introduced in [17]. Indeed, the propositional fragment of FML_n [17] is obtained from BML_n

by deleting the single-succedent and -antecedent conditions in the logical inference rules discussed above, and modifying the inference rules concerning $\sim_k\sim_j$.

4. BML_n can be seen as an extension of the Gentzen-type sequent calculus BL introduced in [20] for bi-intuitionistic logic. BL is an extension of Takeuti's sequent calculus LJ′ presented in [35] for intuitionistic logic.
5. Since the cut-elimination theorem does not hold for BL [20], the cut-elimination theorem also does not hold for BML_n.
6. The sequents of the form $\alpha \Rightarrow \alpha$ for any formula α are provable in BML_n. This fact can be shown by induction on α.
7. The following sequents are provable in BML_n:

   (a) $\sim_j(\alpha{\rightarrow_j}\beta) \Leftrightarrow \sim_j\beta{\leftarrow_j}\sim_j\alpha$,
   (b) $\sim_j(\alpha{\leftarrow_j}\beta) \Leftrightarrow \sim_j\beta{\rightarrow_j}\sim_j\alpha$,
   (c) $\sim_k(\alpha{\rightarrow_j}\beta) \Leftrightarrow \sim_k\alpha{\rightarrow_j}\sim_k\beta$,
   (d) $\sim_k(\alpha{\leftarrow_j}\beta) \Leftrightarrow \sim_k\alpha{\leftarrow_j}\sim_k\beta$,

   thus elucidating the role of condition $j \neq k$ in the rules for negated implications and co-implications.
8. Some Gentzen-type sequent calculi that can prove the sequents $\sim(\alpha{\rightarrow}\beta) \Leftrightarrow \sim\beta{\leftarrow}\sim\alpha$ and $\sim(\alpha{\leftarrow}\beta) \Leftrightarrow \sim\beta{\rightarrow}\sim\alpha$, analogous to the sequents presented just above, were studied in [18,38].
9. The rules ($\sim_j\sim_k$left), ($\sim_j\sim_k$right), ($\sim_k\sim_j$left) and ($\sim_k\sim_j$right) demonstrate how the combinations of $\sim_j$ and $\sim_k$ can be used to model the connectives of intuitionistic negation and dual-intuitionistic negation (co-negation). In fact, the following sequents are easily derivable with $j < k$:

   (a) $\sim_j\sim_k\alpha \Leftrightarrow \alpha \rightarrow_k (\alpha \leftarrow_k \alpha)$,
   (b) $\sim_k\sim_j\alpha \Leftrightarrow (\alpha \rightarrow_k \alpha) \leftarrow_k \alpha$.

   By way of illustration we present here the derivation of (a), the derivation of (b) being analogous:

$$\dfrac{\dfrac{\alpha \Rightarrow \alpha \quad \dfrac{\alpha \Rightarrow \alpha}{\alpha, \sim_j\sim_k\alpha \Rightarrow}\ (\sim_j\sim_k\text{left})}{\alpha, \sim_j\sim_k\alpha \Rightarrow \alpha \leftarrow_k \alpha}\ (\leftarrow_k\text{right})}{\sim_j\sim_k\alpha \Rightarrow \alpha \rightarrow_k (\alpha \leftarrow_k \alpha)}\ (\rightarrow_k\text{right})$$

$$\dfrac{\dfrac{\alpha \Rightarrow \alpha \quad \dfrac{\alpha \Rightarrow \alpha}{\alpha \leftarrow_k \alpha \Rightarrow}\ (\leftarrow_k\text{left})}{\alpha, \alpha \rightarrow_k (\alpha \leftarrow_k \alpha) \Rightarrow}\ (\rightarrow_k\text{left})}{\alpha \rightarrow_k (\alpha \leftarrow_k \alpha) \Rightarrow \sim_j\sim_k\alpha}\ (\sim_j\sim_k\text{right})$$

Now, for any k, $\alpha \leftarrow_k \alpha$ can be naturally interpreted as a *falsum* constant ($\bot_k$), and $\alpha \rightarrow_k \alpha$ as a *verum* constant ($\top_k$). This means that $\sim_j\sim_k$ and $\sim_k\sim_j$ (with $j < k$) indeed have the characteristic properties of intuitionistic and dual-intuitionistic negations correspondingly. It is to be noted that condition $j < k$ plays here an important role, serving as an effective technical device for distinguishing between intuitionistic and dual-intuitionistic negations.

In [20] the Gentzen-type sequent calculus BL for bi-intuitionistic logic is introduced as formulated below. Formulas of BL are constructed from countably many propositional variables by the logical connectives $\wedge$, $\vee$, $\rightarrow$ and $\leftarrow$.

DEFINITION 2.2. (BL) The initial sequents of BL are of the following form, for any propositional variable p,

$$p \Rightarrow p.$$

The structural inference rules of BL are the same as those of BML_n.

The logical inference rules of BL are of the form:

$$\frac{\Gamma \Rightarrow \Delta, \alpha \quad \beta, \Sigma \Rightarrow \Pi}{\alpha{\rightarrow}\beta, \Gamma, \Sigma \Rightarrow \Delta, \Pi}\ (\rightarrow\text{left}) \qquad \frac{\alpha, \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha{\rightarrow}\beta}\ (\rightarrow\text{right})$$

$$\frac{\alpha \Rightarrow \Delta, \beta}{\alpha{\leftarrow}\beta \Rightarrow \Delta}\ (\leftarrow\text{left}) \qquad \frac{\Gamma \Rightarrow \Delta, \alpha \quad \beta, \Sigma \Rightarrow \Pi}{\Gamma, \Sigma \Rightarrow \Delta, \Pi, \alpha{\leftarrow}\beta}\ (\leftarrow\text{right})$$

$$\frac{\alpha, \beta, \Gamma \Rightarrow \Delta}{\alpha \wedge \beta, \Gamma \Rightarrow \Delta}\ (\wedge\text{left}) \qquad \frac{\Gamma \Rightarrow \Delta, \alpha \quad \Gamma \Rightarrow \Delta, \beta}{\Gamma \Rightarrow \Delta, \alpha \wedge \beta}\ (\wedge\text{right})$$

$$\frac{\alpha, \Gamma \Rightarrow \Delta \quad \beta, \Gamma \Rightarrow \Delta}{\alpha \vee \beta, \Gamma \Rightarrow \Delta}\ (\vee\text{left}) \qquad \frac{\Gamma \Rightarrow \Delta, \alpha, \beta}{\Gamma \Rightarrow \Delta, \alpha \vee \beta}\ (\vee\text{right}).$$

Some remarks on BL may he helpful.

1. The sequents of the form $\alpha \Rightarrow \alpha$ for any formula α are provable in BL. This fact can be shown by induction on α.
2. The $\leftarrow$-free fragment of BL extended by the inference rules for intuitionistic negation is equivalent to Takeuti's sequent calculus LJ′ presented in [35] for intuitionistic logic.
3. The cut-elimination theorem does not hold for BL, but holds for LJ′.
4. Intuitionistic negation ($\neg_{in}$) and co-negation ($\neg_{co}$) can be introduced in BL through the constants $\top$ and $\bot$, defined as $p \rightarrow p$ and $p \leftarrow p$

correspondingly, where p is some fixed propositional variable. We have then: $\neg_{in}\alpha := \alpha \to \bot$, and $\neg_{co}\alpha := \top \leftarrow \alpha$.

5. The following inference rules are derivable in BL for $\neg_{in}$ and $\neg_{co}$ so defined:

$$\frac{\Gamma \Rightarrow \alpha}{\neg_{in}\alpha, \Gamma \Rightarrow}\ (\neg_{in}\text{left}) \qquad \frac{\alpha, \Gamma \Rightarrow}{\Gamma \Rightarrow \neg_{in}\alpha}\ (\neg_{in}\text{right})$$

$$\frac{\Rightarrow \Gamma, \alpha}{\neg_{co}\alpha \Rightarrow \Gamma}\ (\neg_{co}\text{left}) \qquad \frac{\alpha \Rightarrow \Gamma}{\Rightarrow \Gamma, \neg_{co}\alpha}\ (\neg_{co}\text{right}).$$

Next, we introduce a Kripke semantics for BML_n.

Definition 2.3. A *Kripke frame* is a structure $\langle M, \leq\rangle$ satisfying the following conditions:

1. M is a nonempty set,
2. $\leq$ is a preorder (i.e., a reflexive and transitive binary relation) on M.

Definition 2.4. Let n (>1) be the positive integer determined by some n-lattice, and j, k be any positive integers with $j, k \leq n$ and $j \neq k$ (and $j < k$ in $\sim_j\sim_k$ and $\sim_k\sim_j$). Let Φ be the set of propositional variables and $\Phi^{\sim}$ be the set $\{\sim_j p \mid p \in \Phi \text{ and } 0 \leq j \leq n\}$ of negated propositional variables. A *paraconsistent valuation* $\models^p$ on a Kripke frame $\langle M, \leq\rangle$ is a mapping from the set $\Phi \cup \Phi^{\sim}$ of literals (i.e., propositional variables and their negations) to the power set 2^M of M such that for any $p \in \Phi$ and any $x, y \in M$, if $x \in\ \models^p (p)$ and $x \leq y$, then $y \in\ \models^p (p)$. We will write $x \models^p p$ for $x \in\ \models^p (p)$. This paraconsistent valuation $\models^p$ is extended to a mapping from the set of all formulas to 2^M by:

1. $x \models^p \alpha{\to_j}\beta$ iff $\forall y \in M$ [$x \leq y$ and $y \models^p \alpha$ imply $y \models^p \beta$],
2. $x \models^p \alpha{\leftarrow_j}\beta$ iff $\exists y \in M$ [$x \geq y$ and $y \models^p \alpha$ and $y \not\models^p \beta$],
3. $x \models^p \alpha{\wedge_j}\beta$ iff $x \models^p \alpha$ and $x \models^p \beta$,
4. $x \models^p \alpha{\vee_j}\beta$ iff $x \models^p \alpha$ or $x \models^p \beta$,
5. $x \models^p {\sim_j}{\sim_j}\alpha$ iff $x \models^p \alpha$,
6. $x \models^p {\sim_j}(\alpha{\to_j}\beta)$ iff $\exists y \in M$ [$x \geq y$ and $y \models^p {\sim_j}\beta$ and $y \not\models^p {\sim_j}\alpha$],
7. $x \models^p {\sim_j}(\alpha{\leftarrow_j}\beta)$ iff $\forall y \in M$ [$x \leq y$ and $y \models^p {\sim_j}\beta$ imply $y \models^p {\sim_j}\alpha$],
8. $x \models^p {\sim_j}(\alpha{\wedge_j}\beta)$ iff $x \models^p {\sim_j}\alpha$ or $x \models^p {\sim_j}\beta$,
9. $x \models^p {\sim_j}(\alpha{\vee_j}\beta)$ iff $x \models^p {\sim_j}\alpha$ and $x \models^p {\sim_j}\beta$,
10. $x \models^p {\sim_k}(\alpha{\to_j}\beta)$ iff $\forall y \in M$ [$x \leq y$ and $y \models^p {\sim_k}\alpha$ imply $y \models^p {\sim_k}\beta$],

11. $x \models^p \sim_k(\alpha \leftarrow_j \beta)$ iff $\exists y \in M$ $[x \geq y$ and $y \models^p \sim_k \alpha$ and $y \not\models^p \sim_k \beta]$,
12. $x \models^p \sim_k(\alpha \wedge_j \beta)$ iff $x \models^p \sim_k \alpha$ and $x \models^p \sim_k \beta$,
13. $x \models^p \sim_k(\alpha \vee_j \beta)$ iff $x \models^p \sim_k \alpha$ or $x \models^p \sim_k \beta$,
14. $x \models^p \sim_j \sim_k \alpha$ iff $\forall y \in M$ $[x \leq y$ implies $y \not\models^p \alpha]$,
15. $x \models^p \sim_k \sim_j \alpha$ iff $\exists y \in M$ $[x \geq y$ and $y \not\models^p \alpha]$.

The following *hereditary condition* holds for $\models^p$: For any formula α and any $x, y \in M$, if $x \models^p \alpha$ and $x \leq y$, then $y \models^p \alpha$.

Definition 2.5. A *paraconsistent Kripke model* is a structure $\langle M, \leq, \models^p \rangle$ such that

1. $\langle M, \leq \rangle$ is a Kripke frame,
2. $\models^p$ is a paraconsistent valuation on $\langle M, \leq \rangle$.

A formula α is *true* in a paraconsistent Kripke model $\langle M, \leq, \models^p \rangle$ iff $x \models^p \alpha$ for all $x \in M$. A formula α is *BML-valid* in a Kripke frame $\langle M, \leq \rangle$ iff it is true for all paraconsistent valuations $\models^p$ on the Kripke frame. A sequent $\Gamma \Rightarrow \Delta$ is BML-valid in a Kripke frame $\langle M, \leq \rangle$ (denoted by $\mathrm{BML}_n \models \Gamma \Rightarrow \Delta$) iff for all paraconsistent valuations $\models^p$ on the Kripke frame, if γ is BML-valid for all $\gamma \in \Gamma$, then δ is BML-valid for some $\delta \in \Delta$.

Next, we present a Kripke semantics for BL [28, 29].

Definition 2.6. A *valuation* $\models$ on a Kripke frame $\langle M, \leq \rangle$ is a mapping from the set Φ of propositional variables to the power set 2^M of M such that for any $p \in \Phi$ and any $x, y \in M$, if $x \in \models(p)$ and $x \leq y$, then $y \in \models(p)$. We will write $x \models p$ for $x \in \models(p)$. This valuation $\models$ is extended to a mapping from the set of all formulas to 2^M by:

1. $x \models \alpha \rightarrow \beta$ iff $\forall y \in M$ $[x \leq y$ and $y \models \alpha$ imply $y \models \beta]$,
2. $x \models \alpha \leftarrow \beta$ iff $\exists y \in M$ $[x \geq y$ and $y \models \alpha$ and $y \not\models \beta]$,
3. $x \models \alpha \wedge \beta$ iff $x \models \alpha$ and $x \models \beta$,
4. $x \models \alpha \vee \beta$ iff $x \models \alpha$ or $x \models \beta$,

The following hereditary condition holds for $\models$: For any formula α and any $x, y \in M$, if $x \models \alpha$ and $x \leq y$, then $y \models \alpha$.

Definition 2.7. A *Kripke model* is a structure $\langle M, \leq, \models \rangle$ such that

1. $\langle M, \leq \rangle$ is a Kripke frame,
2. $\models$ is a valuation on $\langle M, \leq \rangle$.

A formula α is *true* in a Kripke model $\langle M, \leq, \models\rangle$ iff $x \models \alpha$ for all $x \in M$. A formula α is *BL-valid* in a Kripke frame $\langle M, \leq\rangle$ iff it is true for all valuations $\models$ on the Kripke frame. A sequent $\Gamma \Rightarrow \Delta$ is BL-valid in a Kripke frame $\langle M, \leq\rangle$ (denoted by BL $\models \Gamma \Rightarrow \Delta$) iff for all valuations $\models$ on the Kripke frame, if γ is BL-valid for all $\gamma \in \Gamma$, then δ is BL-valid for some $\delta \in \Delta$.

The following completeness theorem for bi-intuitionistic logic is known [28,29]: For any finite sets Γ and Δ of formulas, BL $\vdash \Gamma \Rightarrow \Delta$ iff BL $\models \Gamma \Rightarrow \Delta$.

3. Mutual Embeddings Between BML$_n$ and BL, and the Completeness Theorem

Firstly, we introduce a translation function f from the formulas of BML$_n$ into those of BL.

Definition 3.1. Let n (>1) be the positive integer determined by some n-lattice, and j, k be any positive integers with $j, k \leq n$ and $j \neq k$. We fix a set Φ of propositional variables and define for every j the sets $\Phi^j := \{p^j \mid p \in \Phi\}$ of propositional variables. The language $\mathcal{L}_{\mathrm{BML}}$ of BML$_n$ is defined using Φ, $\wedge_j, \vee_j$, $\rightarrow_j$, $\leftarrow_j$ and $\sim_j$. The language $\mathcal{L}_{\mathrm{BL}}$ of BL is defined using Φ, $\Phi^1, \ldots, \Phi^n$, $\wedge$, $\vee$, $\rightarrow$, $\leftarrow$. (Moreover, $\neg_{in}$ and $\neg_{co}$ can be introduced by definitions given in remark 4 on page 9.) A mapping f from $\mathcal{L}_{\mathrm{BML}}$ to $\mathcal{L}_{\mathrm{BL}}$ is defined by:

1. For any $p \in \Phi$, for any $j \leq n$, $f(p) := p$ and $f(\sim_j p) := p^j \in \Phi^j$,
2. $f(\alpha{\rightarrow_j}\beta) := f(\alpha){\rightarrow}f(\beta)$,
3. $f(\alpha{\leftarrow_j}\beta) := f(\alpha){\leftarrow}f(\beta)$,
4. $f(\alpha{\wedge_j}\beta) := f(\alpha) \wedge f(\beta)$,
5. $f(\alpha{\vee_j}\beta) := f(\alpha) \vee f(\beta)$,
6. $f(\sim_j\sim_j\alpha) := f(\alpha)$,
7. $f(\sim_j(\alpha{\rightarrow_j}\beta)) := f(\sim_j\beta){\leftarrow}f(\sim_j\alpha)$,
8. $f(\sim_j(\alpha{\leftarrow_j}\beta)) := f(\sim_j\beta){\rightarrow}f(\sim_j\alpha)$,
9. $f(\sim_j(\alpha{\wedge_j}\beta)) := f(\sim_j\alpha) \vee f(\sim_j\beta)$,
10. $f(\sim_j(\alpha{\vee_j}\beta)) := f(\sim_j\alpha) \wedge f(\sim_j\beta)$,
11. $f(\sim_k(\alpha{\rightarrow_j}\beta)) := f(\sim_k\alpha){\rightarrow}f(\sim_k\beta)$,
12. $f(\sim_k(\alpha{\leftarrow_j}\beta)) := f(\sim_k\alpha){\leftarrow}f(\sim_k\beta)$,
13. $f(\sim_k(\alpha{\wedge_j}\beta)) := f(\sim_k\alpha) \wedge f(\sim_k\beta)$,

14. $f({\sim}_k(\alpha{\vee}_j\beta)) := f({\sim}_k\alpha) \vee f({\sim}_k\beta)$,
15. $f({\sim}_j{\sim}_k\alpha) := \neg_{in} f(\alpha)$, provided $j < k$,
16. $f({\sim}_k{\sim}_j\alpha) := \neg_{co} f(\alpha)$, provided $j < k$.

An expression $f(\Gamma)$ denotes the result of replacing every occurrence of a formula α in Γ by an occurrence of $f(\alpha)$. Analogous notation is used for the other mappings discussed later.

Some remarks are in place.

1. The translation function f is formally the same as the translation function introduced in [17] for classical multilattice logic.
2. A similar translation has been used by Gurevich [14], Rautenberg [30] and Vorob'ev [36] to embed Nelson's three-valued constructive logic [1, 24] into intuitionistic logic.
3. Some similar translations have also been used, for example, in [15,16] to embed some paraconsistent logics into classical logic or intuitionistic logic.

Next, we introduce a translation function g from the formulas of BL into those of BML_n.

Definition 3.2. Let $\mathcal{L}_{\text{BML}}$ and $\mathcal{L}_{\text{BL}}$ be the languages defined in Definition 3.1.

A mapping g from $\mathcal{L}_{\text{BL}}$ to $\mathcal{L}_{\text{BML}}$ is defined by:

1. For any $j \leq n$, any $p \in \Phi$, and any $p^j \in \Phi^j$, $g(p) := p$ and $g(p^j) := {\sim}_j p$;
2. $g(\alpha{\rightarrow}\beta) := g(\alpha){\rightarrow}_j g(\beta)$, where j is a fixed positive integer, such that $j \leq n$;
3. $g(\alpha{\leftarrow}\beta) := g(\alpha){\leftarrow}_j g(\beta)$, where j is a fixed positive integer, such that $j \leq n$;
4. $g(\alpha \wedge \beta) := g(\alpha){\wedge}_j g(\beta)$, where j is a fixed positive integer, such that $j \leq n$;
5. $g(\alpha \vee \beta) := g(\alpha){\vee}_j g(\beta)$, where j is a fixed positive integer, such that $j \leq n$.

Note that translations for $\neg_{in}\alpha$ and $\neg_{co}\alpha$ can be obtained from their definitions as $g(\neg_{in}\alpha) := {\sim}_j{\sim}_k g(\alpha)$ and $g(\neg_{co}\alpha) := {\sim}_k{\sim}_j g(\alpha)$, where j and k are two fixed positive integers, such that $j, k \leq n$ and $j < k$.

We can obtain the following syntactical embedding theorems in a similar way as in [17].

Theorem 3.3. (Syntactical embedding from BML_n into BL) *Let* Γ, Δ *be sets of formulas in* $\mathcal{L}_{\mathrm{BML}}$*, and* f *be the mapping defined in Definition* 3.1. $\mathrm{BML}_n \vdash \Gamma \Rightarrow \Delta$ *iff* $\mathrm{BL} \vdash f(\Gamma) \Rightarrow f(\Delta)$.

Proof. • ($\Longrightarrow$): By induction on the proofs P of $\Gamma \Rightarrow \Delta$ in BML_n. We show some cases.

1. Case ($\sim_j \!\rightarrow_j$left). The last inference of P is of the form:

$$\frac{\sim_j \beta \Rightarrow \Delta, \sim_j \alpha}{\sim_j(\alpha \rightarrow_j \beta) \Rightarrow \Delta}\ (\sim_j \!\rightarrow_j\text{left}).$$

By induction hypothesis, we have $\mathrm{BL} \vdash f(\sim_j \beta) \Rightarrow f(\Delta), f(\sim_j \alpha)$. Then, we obtain the required fact:

$$\frac{\begin{matrix} \vdots \\ f(\sim_j \beta) \Rightarrow f(\Delta), f(\sim_j \alpha) \end{matrix}}{f(\sim_j \beta) \leftarrow f(\sim_j \alpha) \Rightarrow f(\Delta)}\ (\leftarrow\text{left}),$$

where $f(\sim_j \beta) \leftarrow f(\sim_j \alpha)$ coincides with $f(\sim_j(\alpha \rightarrow_j \beta))$ by the definition of f.

2. Case ($\sim_k \!\rightarrow_j$right). The last inference of P is of the form:

$$\frac{\sim_k \alpha, \Gamma \Rightarrow \sim_k \beta}{\Gamma \Rightarrow \sim_k(\alpha \rightarrow_j \beta)}\ (\sim_k \!\rightarrow_j\text{right}).$$

By induction hypothesis, we have $\mathrm{BL} \vdash f(\sim_k \alpha), f(\Gamma) \Rightarrow f(\sim_k \beta)$. Then, we obtain the required fact:

$$\frac{\begin{matrix} \vdots \\ f(\sim_k \alpha), f(\Gamma) \Rightarrow f(\sim_k \beta) \end{matrix}}{f(\Gamma) \Rightarrow f(\sim_k \alpha) \rightarrow f(\sim_k \beta)}\ (\rightarrow\text{right}),$$

where $f(\sim_k \alpha) \rightarrow f(\sim_k \beta)$ coincides with $f(\sim_k(\alpha \rightarrow_j \beta))$ by the definition of f.

3. Case ($\sim_j \!\leftarrow_j$left). The last inference of P is of the form:

$$\frac{\Gamma \Rightarrow \Delta, \sim_j \beta \quad \sim_j \alpha, \Sigma \Rightarrow \Pi}{\sim_j(\alpha \leftarrow_j \beta), \Gamma, \Sigma \Rightarrow \Delta, \Pi}\ (\sim_j \!\leftarrow_j\text{left})$$

By induction hypothesis, we have:

$\mathrm{BL} \vdash f(\Gamma) \Rightarrow f(\Delta), f(\sim_j \beta)$ and $\mathrm{BL} \vdash f(\sim_j \alpha), f(\Sigma) \Rightarrow f(\Pi)$. Then, we obtain the required fact:

$$\dfrac{\begin{array}{c}\vdots\\ f(\Gamma) \Rightarrow f(\Delta), f(\sim_j\beta)\end{array} \quad \begin{array}{c}\vdots\\ f(\sim_j\alpha), f(\Sigma) \Rightarrow f(\Pi)\end{array}}{f(\sim_j\alpha){\rightarrow}f(\sim_j\beta), f(\Gamma), f(\Sigma) \Rightarrow f(\Delta), f(\Pi)}\ (\rightarrow\text{left}),$$

where $f(\sim_j\beta){\rightarrow}f(\sim_j\alpha)$ coincides with $\sim_j(\alpha{\leftarrow_j}\beta)$ by the definition of f.

4. Case ($\sim_k{\leftarrow_j}$right). The last inference of P is of the form:

$$\dfrac{\Gamma \Rightarrow \Delta, \sim_k\alpha \quad \sim_k\beta, \Sigma \Rightarrow \Pi}{\Gamma, \Sigma \Rightarrow \Delta, \Pi, \sim_k(\alpha{\leftarrow_j}\beta)}\ (\sim_k{\leftarrow_j}\text{right})$$

By induction hypothesis, we have:
BL $\vdash f(\Gamma) \Rightarrow f(\Delta), f(\sim_k\alpha)$ and $f(\sim_k\beta), f(\Sigma) \Rightarrow f(\Pi)$. Then, we obtain the required fact:

$$\dfrac{\begin{array}{c}\vdots\\ f(\Gamma) \Rightarrow f(\Delta), f(\sim_k\alpha))\end{array} \quad \begin{array}{c}\vdots\\ f(\sim_k\beta), f(\Sigma) \Rightarrow f(\Pi)\end{array}}{f(\Gamma), f(\Sigma) \Rightarrow f(\Delta), f(\Pi), f(\sim_k\alpha){\leftarrow}f(\sim_k\beta)}\ (\leftarrow\text{right}),$$

where $f(\sim_k\alpha){\leftarrow}f(\sim_k\beta)$ coincides with $\sim_k(\alpha{\leftarrow_j}\beta)$ by the definition of f.

- ($\Longleftarrow$): By induction on the proofs Q of $f(\Gamma) \Rightarrow f(\Delta)$ in BL.

1. Case (cut). The last inference of Q is of the form:

$$\dfrac{f(\Gamma) \Rightarrow f(\Delta), \beta \quad \beta, f(\Sigma) \Rightarrow f(\Pi)}{f(\Gamma), f(\Sigma) \Rightarrow f(\Delta), f(\Pi)}\ (\text{cut}).$$

In this case, β is a formula of BL. We then have the fact $\gamma = f(\gamma)$ for any formula γ in BL. This can be shown by induction on γ. Thus, Q is of the form:

$$\dfrac{f(\Gamma) \Rightarrow f(\Delta), f(\beta) \quad f(\beta), f(\Sigma) \Rightarrow f(\Pi)}{f(\Gamma), f(\Sigma) \Rightarrow f(\Delta), f(\Pi)}\ (\text{cut}).$$

By induction hypothesis, we have $\text{BML}_n \vdash \Gamma \Rightarrow \Delta, \beta$ and $\text{BML}_n \vdash \beta, \Sigma \Rightarrow \Pi$. Then, we obtain the required fact:

$$\dfrac{\begin{array}{c}\vdots\\ \Gamma \Rightarrow \Delta, \beta\end{array} \quad \begin{array}{c}\vdots\\ \beta, \Sigma \Rightarrow \Pi\end{array}}{\Gamma, \Sigma \Rightarrow \Delta, \Pi}\ (\text{cut}).$$

2. Case ($\leftarrow$left). The last inference of Q is of the form:

$$\dfrac{f(\alpha) \Rightarrow f(\Delta), f(\beta)}{f(\alpha{\leftarrow_j}\beta) \Rightarrow f(\Delta)}\ (\leftarrow\text{left}),$$

where $f(\alpha{\leftarrow_j}\beta)$ coincides with $f(\alpha){\leftarrow}f(\beta))$ by the definition of f. By induction hypothesis we have $\mathrm{BML}_n \vdash \alpha \Rightarrow \Delta, \beta$. Then, we obtain the required fact:

$$\dfrac{\begin{array}{c}\vdots\\ \alpha \Rightarrow \Delta, \beta\end{array}}{\alpha{\leftarrow_j}\beta \Rightarrow \Delta}\ ({\leftarrow_j}\text{left}).$$

∎

By an analogous induction procedure we get also the following theorem:

Theorem 3.4. (Syntactical embedding from BL into BML_n) *Let Γ, Δ be sets of formulas in $\mathcal{L}_{\mathrm{BL}}$, and g be the mapping defined in Definition 3.2.*
$\mathrm{BL} \vdash \Gamma \Rightarrow \Delta$ *iff* $\mathrm{BML}_n \vdash g(\Gamma) \Rightarrow g(\Delta)$.

Next, we show the semantical embedding theorem of BML_n into BL.

Lemma 3.5. *Let f be the mapping defined in Definition 3.1. For any paraconsistent Kripke model $\langle M, \leq, \models^p\rangle$, we can construct a Kripke model $\langle M, \leq, \models\rangle$ such that for any formula α and any $x \in M$,*

$x \models^p \alpha$ *iff* $x \models f(\alpha)$.

Proof. Let Φ be a set of propositional variables, and for every positive integer $j \leq n$ let Φ^j be the set $\{p^j \mid p \in \Phi\}$ of propositional variables. Let $\Phi^{\sim}$ be the set $\{\sim_j p \mid p \in \Phi \text{ and } 0 \leq j \leq n\}$ of negated propositional variables. Suppose that $\langle M, \leq, \models^p\rangle$ is a paraconsistent Kripke model where $\models^p$ is a mapping from $\Phi \cup \Phi^{\sim}$ to the power set 2^M of M, and that the hereditary condition with respect to $p \in \Phi \cup \Phi^{\sim}$ holds for $\models^p$. Suppose that $\langle M, \leq, \models\rangle$ is a Kripke model where $\models$ is a mapping from $\Phi \cup \bigcup_{1\leq j\leq n} \Phi^j$ to 2^M, and that the hereditary condition with respect to $p \in \Phi \cup \bigcup_{1\leq j\leq n} \Phi^j$ holds for $\models$. Suppose, moreover, that these models satisfy the following conditions: For any $x \in M$ and any $p \in \Phi$,

1. $x \models^p p$ iff $x \models p$,
2. $x \models^p \sim_j p$ iff $x \models p^j$.

Then, the lemma is proved by induction on α.

• Base step:

1. Case $\alpha \equiv p$ where p is a propositional variable: $x \models^p p$ iff $x \models p$ (by the assumption) iff $x \models f(p)$ (by the definition of f).
2. Case $\alpha \equiv \sim_j p$ where p is a propositional variable: $x \models^p \sim_j p$ iff $x \models p^j$ (by the assumption) iff $x \models f(\sim_j p)$ (by the definition of f).

• Induction step: We show some cases.

1. Case $\alpha \equiv \beta{\rightarrow_j}\gamma$: We obtain: $x \models^p \beta{\rightarrow_j}\gamma$ iff $\forall y \in M$ [$x \leq y$ and $y \models^p \beta$ imply $y \models^p \gamma$] iff $\forall y \in M$ [$x \leq y$ and $y \models f(\beta)$ imply $y \models f(\gamma)$] (by induction hypothesis) iff $x \models f(\beta){\rightarrow}f(\gamma)$ iff $x \models f(\beta{\rightarrow_j}\gamma)$ (by the definition of f).
2. Case $\alpha \equiv \beta{\leftarrow_j}\gamma$: We obtain: $x \models^p \beta{\leftarrow_j}\gamma$ iff $\exists y \in M$ [$y \leq x$ and $y \models^p \beta$ and $y \not\models^p \gamma$] iff $\exists y \in M$ [$y \leq x$ and $y \models f(\beta)$ and $y \not\models f(\gamma)$] (by induction hypothesis) iff $x \models f(\beta){\leftarrow}f(\gamma)$ iff $x \models f(\beta{\leftarrow_j}\gamma)$ (by the definition of f).
3. Case $\alpha \equiv \beta{\wedge_j}\gamma$: We obtain: $x \models^p \beta{\wedge_j}\gamma$ iff $x \models^p \beta$ and $x \models^p \gamma$ iff $x \models f(\beta)$ and $x \models f(\gamma)$ (by induction hypothesis) iff $x \models f(\beta) \wedge f(\gamma)$ iff $x \models f(\beta{\wedge_j}\gamma)$ (by the definition of f).
4. Case $\alpha \equiv {\sim_j}{\sim_j}\beta$: We obtain: $x \models^p {\sim_j}{\sim_j}\beta$ iff $x \models^p \beta$ iff $x \models f(\beta)$ (by induction hypothesis) iff $x \models f({\sim_j}{\sim_j}\beta)$ (by the definition of f).
5. Case $\alpha \equiv {\sim_j}(\beta{\rightarrow_j}\gamma)$: We obtain: $x \models^p {\sim_j}(\beta{\rightarrow_j}\gamma)$ iff $\exists y \in M$ [$y \leq x$ and $y \models^p {\sim_j}\gamma$ and $y \not\models^p {\sim_j}\beta$] iff $\exists y \in M$ [$y \leq x$ and $y \models f({\sim_j}\gamma)$ and $y \not\models f({\sim_j}\beta)$] (by induction hypothesis) iff $x \models f({\sim_j}\gamma){\leftarrow}f({\sim_j}\beta)$ iff $x \models f({\sim_j}(\beta{\rightarrow_j}\gamma))$ (by the definition of f).
6. Case $\alpha \equiv {\sim_j}(\beta{\leftarrow_j}\gamma)$: We obtain: $x \models^p {\sim_j}(\beta{\leftarrow_j}\gamma)$ iff $\forall y \in M$ [$x \leq y$ and $y \models^p {\sim_j}\gamma$ imply $y \models^p {\sim_j}\beta$] iff $\forall y \in M$ [$x \leq y$ and $y \models f({\sim_j}\gamma)$ imply $y \models f({\sim_j}\beta)$] (by induction hypothesis) iff $x \models f({\sim_j}\gamma){\rightarrow}f({\sim_j}\beta)$ iff $x \models f({\sim_j}(\beta{\leftarrow_j}\gamma))$ (by the definition of f).
7. Case $\alpha \equiv {\sim_j}(\beta{\wedge_j}\gamma)$: We obtain: $x \models^p {\sim_j}(\beta{\wedge_j}\gamma)$ iff $x \models^p {\sim_j}\beta$ or $x \models^p {\sim_j}\gamma$ iff $x \models f({\sim_j}\beta)$ or $x \models f({\sim_j}\gamma)$ (by induction hypothesis) iff $x \models f({\sim_j}\beta) \vee f({\sim_j}\gamma)$ iff $x \models f({\sim_j}(\beta{\wedge_j}\gamma))$ (by the definition of f).
8. Case $\alpha \equiv {\sim_k}(\beta{\rightarrow_j}\gamma)$: We obtain: $x \models^p {\sim_k}(\beta{\rightarrow_j}\gamma)$ iff $\forall y \in M$ [$x \leq y$ and $y \models^p {\sim_k}\beta$ imply $y \models^p {\sim_k}\gamma$] iff $\forall y \in M$ [$x \leq y$ and $y \models f({\sim_k}\beta)$ imply $y \models f({\sim_k}\gamma)$] (by induction hypothesis) iff $x \models f({\sim_k}\beta){\rightarrow}f({\sim_k}\gamma)$ iff $x \models f({\sim_k}(\beta{\rightarrow_j}\gamma))$ (by the definition of f).
9. Case $\alpha \equiv {\sim_k}(\beta{\leftarrow_j}\gamma)$: We obtain: $x \models^p {\sim_k}(\beta{\leftarrow_j}\gamma)$ iff $\exists y \in M$ [$y \leq x$ and $y \models^p {\sim_k}\beta$ and $y \not\models^p {\sim_k}\gamma)$] iff $\exists y \in M$ [$y \leq x$ and $y \models f({\sim_k}\beta)$ and $y \not\models f({\sim_k}\gamma)$] (by induction hypothesis) iff $x \models f({\sim_k}\beta){\leftarrow}f({\sim_k}\gamma)$ iff $x \models f({\sim_k}(\beta{\leftarrow_j}\gamma))$ (by the definition of f).
10. Case $\alpha \equiv {\sim_k}(\beta{\wedge_j}\gamma)$: We obtain: $x \models^p {\sim_k}(\beta{\wedge_j}\gamma)$ iff $x \models^p {\sim_k}\beta$ and $x \models^p {\sim_k}\gamma$ iff $x \models f({\sim_k}\beta)$ and $x \models f({\sim_k}\gamma)$ (by induction hypothesis) iff $x \models f({\sim_k}\beta) \wedge f({\sim_k}\gamma)$ iff $x \models f({\sim_k}(\beta{\wedge_j}\gamma))$ (by the definition of f).

11. Case $\alpha \equiv \sim_j\sim_k\beta$, where $j < k$: We obtain: $x \models^p \sim_j\sim_k\beta$ iff $\forall y \in M$ $[x \leq y$ implies $(y \not\models^p \beta)]$ iff $\forall y \in M$ $[x \leq y$ implies $(y \not\models f(\beta))]$ (by induction hypothesis) iff $x \models \neg_{in} f(\beta)$ iff $x \models f(\sim_j\sim_k\beta)$ (by the definition of f).
12. Case $\alpha \equiv \sim_k\sim_j\beta$, where $j < k$: We obtain: $x \models^p \sim_k\sim_j\beta$ iff $\exists y \in M$ $[x \geq y$ and $y \not\models^p \beta]$ iff $\exists y \in M$ $[x \geq y$ and $(y \not\models f(\beta))]$ (by induction hypothesis) iff $x \models \neg_{co} f(\beta)$ iff $x \models f(\sim_k\sim_j\beta)$ (by the definition of f).

■

Lemma 3.6. *Let f be the mapping defined in Definition 3.1. For any Kripke model $\langle M, \leq, \models \rangle$, we can construct a paraconsistent Kripke model $\langle M, \leq, \models^p \rangle$ such that for any formula α and any $x \in M$, $x \models f(\alpha)$ iff $x \models^p \alpha$.*

Proof. Similar to the proof of Lemma 3.5. ■

Theorem 3.7. (Semantical embedding from BML_n into BL) *Let Γ and Δ be sets of formulas in $\mathcal{L}_{\mathrm{BML}}$, and f be the mapping defined in Definition 3.1.* $\mathrm{BML}_n \models \Gamma \Rightarrow \Delta$ *iff* $\mathrm{BL} \models f(\Gamma) \Rightarrow f(\Delta)$.

Proof. ($\Longrightarrow$): Assume that if $x \models^p \alpha$ for every $\alpha \in \Gamma$, then $x \models^p \beta$ for some $\beta \in \Delta$. Let $x \models f(\alpha)$ for every $f(\alpha) \in f(\Gamma)$. By Lemma 3.6, we can construct a paraconsistent valuation $\models^p$, such that $x \models^p \alpha$ for any $\alpha \in \Gamma$. Thus, $x \models^p \beta$ for some $\beta \in \Delta$, and by Lemma 3.5, $x \models f(\beta)$ for some $f(\beta) \in f(\Delta)$. ($\Longleftarrow$): Similarly. ■

Next, we show the semantical embedding theorem of BL into BML_n.

Lemma 3.8. *Let g be the mapping defined in Definition 3.2. For any Kripke model $\langle M, \leq, \models \rangle$, we can construct a paraconsistent Kripke model $\langle M, \leq, \models^p \rangle$ such that for any formula α and any $x \in M$, $x \models \alpha$ iff $x \models^p g(\alpha)$.*

Proof. Let Φ be a set of propositional variables, and for every positive integer $j \leq n$ let Φ^j be the set $\{p^j \mid p \in \Phi\}$ of propositional variables. Let $\Phi^\sim$ be the set $\{\sim_j p \mid p \in \Phi$ and $0 \leq j \leq n\}$ of negated propositional variables. Suppose that $\langle M, \leq, \models^p \rangle$ is a paraconsistent Kripke model where $\models^p$ is a mapping from $\Phi \cup \Phi^\sim$ to the power set 2^M of M, and that the hereditary condition with respect to $p \in \Phi \cup \Phi^\sim$ holds for $\models^p$. Suppose that $\langle M, \leq, \models \rangle$ is a Kripke model where $\models$ is a mapping from $\Phi \cup \bigcup_{1 \leq j \leq n} \Phi^j$ to 2^M, and that the hereditary condition with respect to $p \in \Phi \cup \bigcup_{1 \leq j \leq n} \Phi^j$ holds for $\models$. Suppose moreover that these models satisfy the following conditions: For any $x \in M$ and any $p \in \Phi$,

1. $x \models^p p$ iff $x \models p$,
2. $x \models^p \sim_j p$ iff $x \models p^j$.

Then, this lemma is proved by induction on α.

• Base step:

1. Case $\alpha \equiv p$ where p is a propositional variable. $x \models p$ iff $x \models^p p$ (by the assumption) iff $x \models^p g(p)$ (by the definition of g).
2. Case $\alpha \equiv p^j$ where p^j is a propositional variable. $x \models p^j$ iff $x \models^p \sim_j p$ (by the assumption) iff $x \models^p g(p^j)$ (by the definition of g).

• Induction step: We show some cases.

1. Case $\alpha \equiv \beta{\rightarrow}\gamma$. We obtain: $x \models \beta{\rightarrow}\gamma$ iff $\forall y \in M$ [$x \leq y$ and $y \models \beta$ imply $y \models \gamma$] iff $\forall y \in M$ [$x \leq y$ and $y \models^p g(\beta)$ imply $y \models^p g(\gamma)$] (by induction hypothesis) iff $x \models^p g(\beta){\rightarrow_j} g(\gamma)$ iff $x \models g(\beta{\rightarrow}\gamma)$ (by the definition of g).
2. Case $\alpha \equiv \beta \wedge \gamma$. We obtain: $x \models \beta \wedge \gamma$ iff $x \models \beta$ and $x \models \gamma$ iff $x \models^p g(\beta)$ and $x \models^p g(\gamma)$ (by induction hypothesis) iff $x \models^p g(\beta) \wedge_j g(\gamma)$ iff $x \models^p g(\beta \wedge \gamma)$ (by the definition of g).
3. Case $\alpha \equiv \beta{\leftarrow}\gamma$. We obtain: $x \models \beta{\leftarrow}\gamma$ iff $\exists y \in M$ [$x \geq y$ and $y \models \beta$ and $y \not\models \gamma$] iff $\exists y \in M$ [$x \geq y$ and $y \models^p g(\beta)$ and $y \not\models^p g(\gamma)$] (by induction hypothesis) iff $x \models^p g(\beta){\leftarrow_j} g(\gamma)$ iff $x \models g(\beta{\leftarrow}\gamma)$ (by the definition of g). ■

Lemma 3.9. *Let g be the mapping defined in Definition 3.2. For any paraconsistent Kripke model $\langle M, \leq, \models^p \rangle$, we can construct a Kripke model $\langle M, \leq, \models \rangle$ such that for any formula α and any $x \in M$, $x \models^p g(\alpha)$ iff $x \models \alpha$.*

Proof. Similar to the proof of Lemma 3.8. ■

Theorem 3.10. (Semantical embedding from BL into BML_n) *Let Γ and Δ be finite sets of formulas in $\mathcal{L}_{\mathrm{BL}}$, and g be the mapping defined in Definition 3.2.* BL $\models \Gamma \Rightarrow \Delta$ *iff* $\mathrm{BML}_n \models g(\Gamma) \Rightarrow g(\Delta)$.

Proof. By Lemmas 3.8 and 3.9. ■

By using Theorems 3.7 and 3.3, we can obtain the following completeness theorem for BML_n.

Theorem 3.11. (Completeness for BML_n) *For any finite sets Γ and Δ of formulas,* $\mathrm{BML}_n \vdash \Gamma \Rightarrow \Delta$ *iff* $\mathrm{BML}_n \models \Gamma \Rightarrow \Delta$.

Proof. We have the following. $\mathrm{BML}_n \models \Gamma \Rightarrow \Delta$ iff $\mathrm{BL} \models f(\Gamma) \Rightarrow f(\Delta)$ (by Theorem 3.7) iff $\mathrm{BL} \vdash f(\Gamma) \Rightarrow f(\Delta)$ (by the completeness theorem for BL) iff $\mathrm{BML}_n \vdash \Gamma \Rightarrow \Delta$ (by Theorem 3.3). ∎

4. A Connexive Variant of Bi-intuitionistic Multilattice Logic

The language of *bi-intuitionistic connexive n-lattice logic* is the same as that of bi-intuitionistic n-lattice logic.

The Gentzen-type sequent calculus CML_n for bi-intuitionistic connexive n-lattice logic is defined as follows.

Definition 4.1 (CML_n). Let n ($>$1) be the positive integer determined by some n-lattice, and j, k be any positive integers with $j, k \leq n$ and $j \neq k$.

CML_n is obtained from BML_n by replacing the logical inference rules ($\sim_j\!\to_j$left), ($\sim_j\!\to_j$right), ($\sim_j\!\leftarrow_j$left) and ($\sim_j\!\leftarrow_j$right) with the logical inference rules of the form:

$$\frac{\Gamma \Rightarrow \Delta, \alpha \quad \sim_j\beta, \Sigma \Rightarrow \Pi}{\sim_j(\alpha\to_j\beta), \Gamma, \Sigma \Rightarrow \Delta, \Pi}\ (\sim_j\!\to_j\mathrm{left}^c) \qquad \frac{\alpha, \Gamma \Rightarrow \sim_j\beta}{\Gamma \Rightarrow \sim_j(\alpha\to_j\beta)}\ (\sim_j\!\to_j\mathrm{right}^c)$$

$$\frac{\sim_j\alpha \Rightarrow \Delta, \beta}{\sim_j(\alpha\leftarrow_j\beta) \Rightarrow \Delta}\ (\sim_j\!\leftarrow_j\mathrm{left}^c) \qquad \frac{\Gamma \Rightarrow \Delta, \sim_j\alpha \quad \beta, \Sigma \Rightarrow \Pi}{\Gamma, \Sigma \Rightarrow \Delta, \Pi, \sim_j(\alpha\leftarrow_j\beta)}\ (\sim_j\!\leftarrow_j\mathrm{right}^c).$$

Some remarks may be helpful.

1. CML_n can be regarded as a modified multilattice logic version of the Gentzen-type sequent calculus BCL for the bi-intuitionistic connexive logic (also called connexive Heyting–Brouwer logic) introduced in [20].
2. Since the cut-elimination theorem does not hold for BCL and BL [20], the cut-elimination theorem also does not hold for CML_n.
3. The sequents of the form $\alpha \Rightarrow \alpha$ for any formula α are provable in CML_n. This fact can be shown by induction on α.
4. The following sequents are provable in CML_n:
   (a) $\sim_j(\alpha\to_j\beta) \Leftrightarrow \alpha\to_j\sim_j\beta$,
   (b) $\sim_j(\alpha\leftarrow_j\beta) \Leftrightarrow \sim_j\alpha\leftarrow_j\beta$,

   which indicates the main difference between BML_n and CML_n.
5. The system BCL introduced in [20] can prove the following sequents, which are similar to the sequents presented just above, and the first ones just correspond to an axiom scheme that is characteristic of connexive logics [4,23,37,40]:

(a) $\sim(\alpha\rightarrow\beta) \Leftrightarrow \alpha\rightarrow\sim\beta$,
(b) $\sim(\alpha\leftarrow\beta) \Leftrightarrow \sim\alpha\leftarrow\beta$.

Next, we introduce a Kripke semantics for CML_n.

DEFINITION 4.2. Let n (>1) be the positive integer determined by some n-lattice, and j, k be any positive integers with $j, k \leq n$ and $j \neq k$. A *connexive valuation* $\models^c$ on a Kripke frame $\langle M, \leq\rangle$ is a mapping from the set Φ of literals to the power set 2^M of M such that for any $p \in \Phi$ and any $x, y \in M$, if $x \in \models^c (p)$ and $x \leq y$, then $y \in \models^c (p)$. We will write $x \models^c p$ for $x \in \models^c (p)$. This connexive valuation $\models^c$ is extended to a mapping from the set of all formulas to 2^M. This extension is obtained from the same conditions of the paraconsistent valuation $\models^p$ by replacing the conditions:

1. $x \models^p \sim_j(\alpha\rightarrow_j\beta)$ iff $\exists y \in M$ $[x \geq y$ and $y \models^p \sim_j\beta$ and $y \not\models^p \sim_j\alpha]$,
2. $x \models^p \sim_j(\alpha\leftarrow_j\beta)$ iff $\forall y \in M$ $[x \leq y$ and $y \models^p \sim_j\beta$ imply $y \models^p \sim_j\alpha]$,

with the following conditions:

1. $x \models^c \sim_j(\alpha\rightarrow_j\beta)$ iff $\forall y \in M$ $[x \leq y$ and $y \models^c \alpha$ imply $y \models^c \sim_j\beta]$,
2. $x \models^c \sim_j(\alpha\leftarrow_j\beta)$ iff $\exists y \in M$ $[x \geq y$ and $y \models^c \sim_j\alpha$ and $y \not\models^c \beta]$.

The hereditary condition holds for $\models^c$. The notion of a *connexive Kripke model* and the CML-validity of a sequent $\Gamma \Rightarrow \Delta$ (denoted by $\mathrm{CML}_n \models \Gamma \Rightarrow \Delta$) can be defined in a similar way as for BML_n.

We introduce a translation function h from the formulas of CML_n into those of BL.

DEFINITION 4.3. Let n (>1) be the positive integer determined by some n-lattice, and j, k be any positive integers with $j, k \leq n$ and $j \neq k$. The language $\mathcal{L}_{\mathrm{CML}}$ of CML_n is the same as $\mathcal{L}_{\mathrm{BML}}$, and the language $\mathcal{L}_{\mathrm{BL}}$ of BL is defined in Definition 3.1.

A mapping h from $\mathcal{L}_{\mathrm{CML}}$ to $\mathcal{L}_{\mathrm{BL}}$ is obtained from the clauses for f in Definition 3.1 by replacing the conditions:

1. $f(\sim_j(\alpha\rightarrow_j\beta)) := f(\sim_j\beta)\leftarrow f(\sim_j\alpha)$,
2. $f(\sim_j(\alpha\leftarrow_j\beta)) := f(\sim_j\beta)\rightarrow f(\sim_j\alpha)$,

with the following conditions:

1. $h(\sim_j(\alpha\rightarrow_j\beta)) := h(\alpha)\rightarrow h(\sim_j\beta)$,
2. $h(\sim_j(\alpha\leftarrow_j\beta)) := h(\sim_j\alpha)\leftarrow h(\beta)$.

Definition 4.4. Let $\mathcal{L}_{\text{CML}}$ and $\mathcal{L}_{\text{BL}}$ be the languages defined in Definition 4.3. A mapping k from $\mathcal{L}_{\text{BL}}$ to $\mathcal{L}_{\text{CML}}$ is defined by the same conditions as those of g defined in Definition 3.2.

We can obtain the following syntactical embedding theorems in a similar way as for BML_n.

Theorem 4.5. (Syntactical embedding from CML_n into BL) *Let* Γ, Δ *be sets of formulas in* $\mathcal{L}_{\text{CML}}$, *and* h *be the mapping defined in Definition* 4.3. $\text{CML}_n \vdash \Gamma \Rightarrow \Delta$ *iff* $\text{BL} \vdash h(\Gamma) \Rightarrow h(\Delta)$.

Proof. We show only some cases for the direction ($\Longrightarrow$) by induction on the proofs P of $\Gamma \Rightarrow \Delta$ in CML_n.

1. Case ($\sim_j\!\rightarrow_j\!\text{right}^c$): The last inference of P is of the form:

$$\frac{\alpha, \Gamma \Rightarrow {\sim_j}\beta}{\Gamma \Rightarrow {\sim_j}(\alpha{\rightarrow_j}\beta)}\ ({\sim_j}{\rightarrow_j}\text{right}^c).$$

   By induction hypothesis, we have $\text{BL} \vdash h(\alpha), h(\Gamma) \Rightarrow h({\sim_j}\beta)$. Then, we obtain the required fact:

$$\frac{\begin{array}{c}\vdots\\ h(\alpha), h(\Gamma) \Rightarrow h({\sim_j}\beta)\end{array}}{h(\Gamma) \Rightarrow h(\alpha){\rightarrow}h({\sim_j}\beta)}\ (\rightarrow\text{right})$$

   where $h(\alpha){\rightarrow}h({\sim_j}\beta)$ coincides with $h({\sim_j}(\alpha{\rightarrow_j}\beta))$ by the definition of h.

2. Case (${\sim_j}{\leftarrow_j}\text{left}^c$): The last inference of P is of the form:

$$\frac{{\sim_j}\alpha \Rightarrow \Delta, \beta}{{\sim_j}(\alpha{\leftarrow_j}\beta) \Rightarrow \Delta}\ ({\sim_j}{\leftarrow_j}\text{left}^c).$$

   By induction hypothesis, we have $\text{BL} \vdash h({\sim_j}\alpha) \Rightarrow h(\Delta), h(\beta)$. Then, we obtain the required fact:

$$\frac{\begin{array}{c}\vdots\\ h({\sim_j}\alpha) \Rightarrow f(\Delta), h(\beta)\end{array}}{h({\sim_j}\alpha){\leftarrow}h(\beta) \Rightarrow h(\Delta)}\ (\leftarrow\text{left})$$

   where $h({\sim_j}\alpha){\leftarrow}h(\beta)$ coincides with $h({\sim_j}(\alpha{\leftarrow_j}\beta))$ by the definition of h. ■

Theorem 4.6. (Syntactical embedding from BL into CML_n) *Let* Γ, Δ *be sets of formulas in* $\mathcal{L}_{\text{BL}}$, *and* k *be the mapping defined in Definition* 4.4. BL $\vdash \Gamma \Rightarrow \Delta$ *iff* $\text{CML}_n \vdash k(\Gamma) \Rightarrow k(\Delta)$.

Next, we show the semantical embedding theorem of CML_n into BL.

Lemma 4.7. *Let h be the mapping defined in Definition 4.3. For any connexive Kripke model $\langle M, \leq, \models^c \rangle$, we can construct a Kripke model $\langle M, \leq, \models \rangle$ such that for any formula α and any $x \in M$, $x \models^c \alpha$ iff $x \models h(\alpha)$.*

Proof. Similar to the proof of Lemma 3.5. Thus, we only illustrate the following cases of the induction step.

1. Case $\alpha \equiv {\sim_j}(\beta \rightarrow_j \gamma)$: We obtain: $x \models^c {\sim_j}(\beta \rightarrow_j \gamma)$ iff $\forall y \in M$ [$x \leq y$ and $y \models^c \beta$ imply $y \models^c {\sim_j}\gamma$] iff $\forall y \in M$ [$x \leq y$ and $y \models h(\beta)$ imply $y \models h({\sim_j}\gamma)$] (by induction hypothesis) iff $x \models h(\beta) \rightarrow h({\sim_j}\gamma)$ iff $x \models h({\sim_j}(\beta \rightarrow_j \gamma))$ (by the definition of h).
2. Case $\alpha \equiv {\sim_j}(\beta \leftarrow_j \gamma)$: We obtain: $x \models^c {\sim_j}(\beta \leftarrow_j \gamma)$ iff $\exists y \in M$ [$y \leq x$ and $y \models^c {\sim_j}\beta$ and $y \not\models^c \gamma$] iff $\exists y \in M$ [$y \leq x$ and $y \models h({\sim_j}\beta)$ and $y \not\models h(\gamma)$] (by induction hypothesis) iff $x \models h({\sim_j}\beta) \leftarrow h(\gamma)$ iff $x \models h({\sim_j}(\beta \leftarrow_j \gamma))$ (by the definition of h). ■

Lemma 4.8. *Let h be the mapping defined in Definition 4.3. For any Kripke model $\langle M, \leq, \models \rangle$, we can construct a connexive Kripke model $\langle M, \leq, \models^c \rangle$ such that for any formula α and any $x \in M$, $x \models h(\alpha)$ iff $x \models^c \alpha$.*

Proof. Similar to the proof of Lemma 4.7. ■

Theorem 4.9. (Semantical embedding from CML_n into BL) *Let Γ and Δ be sets of formulas in $\mathcal{L}_{\mathrm{CML}}$, and h be the mapping defined in Definition 4.3.* $\mathrm{CML}_n \models \Gamma \Rightarrow \Delta$ *iff* $\mathrm{BL} \models h(\Gamma) \Rightarrow h(\Delta)$.

Proof. Similar to the proof of Theorem 3.7. We use Lemmas 4.7 and 4.8. ■

We can also show the semantical embedding theorem of BL into CML_n.

Theorem 4.10. (Semantical embedding from BL into CML_n) *Let Γ and Δ be sets of formulas in $\mathcal{L}_{\mathrm{BL}}$, and k be the mapping defined in Definition 4.4.* $\mathrm{BL} \models \Gamma \Rightarrow \Delta$ *iff* $\mathrm{CML}_n \models k(\Gamma) \Rightarrow k(\Delta)$.

Proof. Similar to Theorem 3.10. ■

By using Theorems 4.5 and 4.9, we can obtain the following completeness theorem for CML_n.

Theorem 4.11. (Completeness for CML_n) *For any finite sets Γ and Δ of formulas,* $\mathrm{CML}_n \vdash \Gamma \Rightarrow \Delta$ *iff* $\mathrm{CML}_n \models \Gamma \Rightarrow \Delta$.

5. Conclusion

It has often been emphasized that Belnap and Dunn's useful four-valued logic FDE is a convincingly motivated basic paracomplete and paraconsistent system of relevance logic. Various ways of extending FDE by an implication connective have been explored, notably there is the distinction between weak and strong implication in the logic of logical bilattices [6]. A rather natural extension of FDE is obtained by adding intuitionistic implication, as is done in Nelson's constructive four-valued logic N4, [1,19]. Once this extension has been implemented, the addition of a dual to intuitionistic implication is another natural move; whilst a connexive variant offers an alternative understanding of negated implications and co-implications. Eventually, in light of the theory of generalized truth values and multi-lattices [34] the definition of multi-consequence systems gives rise to a combination BML_n of several first-degree entailment systems within a single formalism. Applications of such systems comprising several consequence relations associated with truth, falsity, information, constructivity, and possibly other notions are still to be explored.

Acknowledgements. We would like to thank the anonymous referee for his or her valuable comments. Norihiro Kamide was partially supported by JSPS KAKENHI Grant (C) JP26330263.

References

[1] Almukdad, A., and D. Nelson, Constructible falsity and inexact predicates, *Journal of Symbolic Logic* 49(1):231–233, 1984.

[2] Anderson, A.R., and N.D. Belnap, Tautological entailments, *Philosophical Studies* 13:9–24, 1962.

[3] Anderson, A.R., and N.D. Belnap, First degree entailments, *Mathematische Annalen* 149:302–319, 1963.

[4] Anderson, A.R., and N.D. Belnap, *Entailment: The Logic of Relevance and Necessity, vol. 1*, Princeton University Press, Princeton, New Jersey, 1975.

[5] Angell, R., A propositional logics with subjunctive conditionals, *Journal of Symbolic Logic* 27:327–343, 1962.

[6] Arieli, O., and A. Avron, Reasoning with logical bilattices, *Journal of Logic, Language and Information* 5:25–63, 1996.

[7] Belnap, N.D., A useful four-valued logic, in G. Epstein and J. M. Dunn (eds.), *Modern Uses of Multiple-Valued Logic*, Dordrecht, Reidel, 1977, pp. 5–37.

[8] Belnap, N.D., How a computer should think, in G. Ryle (ed.), *Contemporary Aspects of Philosophy*, Oriel Press, Stocksfield, 1977, pp. 30–56.

[9] Dunn, J.M., Intuitive semantics for first-degree entailment and ‘coupled trees’, *Philosophical Studies* 29(3):149–168, 1976.

[10] Fitting, M., Bilattices are nice things, in T. Bolander, V. Hendricks, and S.A. Pedersen (eds.), *Self-reference*, CSLI Publications, Stanford, 2006, pp. 53–77.

[11] Ginsberg, M., Multi-valued logics, *Proceedings of AAAI-86, Fifth National Conference on Artificial Intelligence*, Morgan Kaufman Publishers, Los Altos, 1986, pp. 243–247.

[12] Ginsberg, M., Multivalued logics: a uniform approach to reasoning in AI, *Computer Intelligence* 4:256–316, 1988.

[13] Goré, R., L. Postniece and A. Tiu, Cut-elimination and proof-search for bi-intuitionistic logic using nested sequents, in C. Areces and R. Goldblatt (eds.), *Advances in Modal Logic*, v. 7, College Publications, 2008, pp. 43–66.

[14] Gurevich, Y., Intuitionistic logic with strong negation, *Studia Logica* 36:49–59, 1977.

[15] Kamide, N., A hierarchy of weak double negations, *Studia Logica* 101(6):1277–1297, 2013.

[16] Kamide, N., Trilattice logic: an embedding-based approach, *Journal of Logic and Computation* 25(3):581–611, 2015.

[17] Kamide, N., and Y. Shramko, Embedding from multilattice logic into classical logic and vice versa, *Journal of Logic and Computation* 27(5):1549–1575, 2017, doi:10.1093/logcom/exw015.

[18] Kamide, N., and H. Wansing, Symmetric and dual paraconsistent logics, *Logic and Logical Philosophy* 19(1-2):7–30, 2010.

[19] Kamide, N., and H. Wansing, *Proof theory of N4-related paraconsistent logics*, Studies in Logic 54, College Publications, London, 2015.

[20] Kamide, N., and H. Wansing, Completeness of connexive Heyting-Brouwer logic, *IFCoLog Journal of Logic and their Applications* 3:441–466, 2016.

[21] Łukowski, P., Modal interpretation of Heyting-Brouwer logic, *Bulletin of the Section of Logic* 25(2):80–83, 1996.

[22] Łukowski, P., A deductive-reductive form of logic: Intuitionistic S4 modalities, *Logic and Logical Philosophy* 10:79–91, 2002.

[23] McCall, S., Connexive implication, *Journal of Symbolic Logic* 31:415–433.

[24] Nelson, D., Constructible falsity, *Journal of Symbolic Logic* 14:16–26, 1949.

[25] Odintsov, S.P., On axiomatizing Shramko-Wansing’s logic, *Studia Logica*, 93:407–428, 2009.

[26] Odintsov, S.P., and H. Wansing, The logic of generalized truth-values and the logic of bilattices, *Studia Logica* 103:91–112, 2015.

[27] Rauszer, C., A formalization of the propositional calculus of H-B logic, *Studia Logica* 33:23–34, 1974.

[28] Rauszer, C., Applications of Kripke models to Heyting-Brouwer logic, *Studia Logica* 36:61–71, 1977.

[29] Rauszer, C., An algebraic and Kripke-style approach to a certain extension of intuitionistic logic, *Dissertations Mathematicae*, Polish Scientific Publishers, Warsaw, 1980, pp. 1–67.

[30] Rautenberg, W., *Klassische und nicht-klassische Aussagenlogik*, Vieweg, Braunschweig, 1979.

[31] Shramko, Y., Truth, falsehood, information and beyond: the American plan generalized, in K. Bimbó (ed.), *J. Michael Dunn on Information Based Logics*, Springer, Dordrecht, 2016, pp. 191–212.

[32] Shramko, Y., J.M. Dunn and T. Takenaka, The trilaticce of constructive truth-values, *Journal of Logic and Computation* 11:761–788, 2001.

[33] Shramko, Y., and H. Wansing, Some useful sixteen-valued logics: How a computer network should think, *Journal of Philosophical Logic* 34:121–153, 2005.

[34] Shramko, Y., and H. Wansing, *Truth and Falsehood. An Inquiry into Generalized Logical Values*, Springer, Dordrecht, 2011.

[35] Takeuti, G., *Proof theory*, North-Holland Publishing Company, Amsterdam, 1975.

[36] Vorob'ev, N.N., A constructive propositional calculus with strong negation (in Russian), *Doklady Akademii Nauk SSSR* 85:465–468, 1952.

[37] Wansing, H., Connexive modal logic, *Advances in Modal Logic* vol. 5, College Publications, London, 2005, pp. 367–385.

[38] Wansing, H., Constructive negation, implication, and co-implication, *Journal of Applied Non-Classical Logics* 18(2-3):341–364, 2008.

[39] Wansing, H., Falsification, natural deduction and bi-intuitionistic logic, *Journal of Logic and Computation* 26(1):425–450, First published online, July 17, 2013.

[40] Wansing, H., Connexive logic, *The Stanford Encyclopedia of Philosophy* (Fall 2014 Edition), E.N. Zalta (ed.), URL = http://plato.stanford.edu/archives/fall2014/entries/logic-connexive/ (First published January 6, 2006).

[41] Wansing, H., On split negation, strong negation, information, falsification, and verification, in K. Bimbó (ed.), *J. Michael Dunn on Information Based Logics*, Springer, Dordrecht, 2016, pp. 161–189.

[42] Wansing, H., Natural deduction for bi-connexive logic and a two-sorted typed λ-calculus, *IFCoLog Journal of Logics and their Applications* 3:413–439, 2016.

[43] Wolter, F., On logics with coimplication, *Journal of Philosophical Logic* 27:353–387, 1998.

N. Kamide
Department of Information and Electronic Engineering
Faculty of Science and Engineering
Teikyo University
Toyosatodai 1-1
Utsunomiya Tochigi 320-8551
Japan
drnkamide08@kpd.biglobe.ne.jp

Y. Shramko
Department of Philosophy
Kryvyi Rih State Pedagogical University
prosp. Gagarina 54
Kryvyi Rih 50086
Ukraine
shramko@rocketmail.com

H. Wansing
Department of Philosophy II
Ruhr-University Bochum
44801 Bochum
Germany
heinrich.wansing@rub.de