Norihiro Kamide and Heinrich Wansing, **Proof Theory of N4-related Paraconsistent Logics**. Studies in Logic vol. 54. College Publications, 2015, pp. 414. ISBN-13: 978-1848901674 (paperback) $20.50.

This book covers a broad spectrum of topics related to the proof-theoretic considerations which have arisen around Nelson's logic of constructible falsity and its paraconsistent versions. It summarizes and in a way organizes some results of the authors' fruitful collaboration in the area of the structural proof theory which evolved over the last decade and which have been reported in a number of previously published papers. Furthermore, the book skilfully combines some general material suitable for a practical reference source on proof systems of various types with the latest achievements of the authors' own important contributions to the field.

The book is divided into four major parts each concentrating on a leading topic such as Nelson's paraconsistent logic, bi-intuitionistic logic in a paraconsistent setting, paraconsistent temporal logics, and paraconsistent substructural logics. Thus, the core theme of the whole work is the idea of paraconsistency in its various dimensions, with a special emphasis on the problem of logical constructivity.

In a short introductory chapter the authors highlight the importance of Nelson's paraconsistent (four-valued) logic N4 as one of the basic and much used deductive instrument in computer science and philosophical logic. It is observed that Belnap and Dunn's four-valued logic and the $\{\wedge, \vee\ \sim\}$-fragment of N4 are coincident, and also that N4 is closely related to bi-intuitionistic logic and dual-intuitionistic logic. Another family of logics relevant to the main topic of the book is focused around the trilattice $SIXTEEN_3$, a ramified algebraic structure, which presents a natural way of generalizing Belnap and Dunn's four-valued logic.

Part I is devoted principally to Nelson's paraconsistent logic itself, and to some "variations" in its vicinity. Several proof systems for the paraconsistent Nelson's logic N4 are thoroughly presented in a form of standard Gentzen calculus, display calculus and natural deduction systems. Some other deductive techniques, such as an auxiliary system, resolution system, subformula calculus and dual calculus are also employed to elucidate certain interesting features of the logics under consideration. The authors also show how N4 can be equipped with a suitable Kripke semantics by using paraconsistent valuations $\models^+$ and $\models^-$, which are not necessarily mutually exhaustive and complementary. It is also demonstrated how formulas of Nelson's logic can be translated into formulas of positive intuitionistic logic. Using these embedding techniques, some basic results concerning N4 including cut-elimination, normalization and completeness are established.

As a side note, by considering the relationships between N4, Belnap and Dunn's four-valued logic and Anderson and Belnap's logic E_{fde} of first-degree entailment (§ 2.3.5), a more detailed explanation on what is precisely understood by the later two logics would be desirable. One might ask here what is the difference (if any)

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between Belnap and Dunn's four-valued logic and Anderson and Belnap's logic E_{fde} of first-degree entailment. The authors are not very specific on this issue, sharing thus an existing tradition of an ambivalent usage of the terms "Belnap's logic" and "first-degree entailment", when these terms are sometimes used as synonyms, and sometimes can refer to quite different logical systems. Strictly speaking, "first degree entailment", as originally introduced by Belnap and deductively formalized by Anderson and Belnap in § 15 of their "Entailment", presents a *binary* consequence system dealing exclusively with consequence expressions of the form $A \to B$, where $\to$ stands for a consequence (entailment) relation between *single* formulas A and B, both containing only $\wedge, \vee$ and $\sim$ (and maybe other truth-functional connectives defined by these). On the other hand, a more general notion of "Belnap's" (or "Belnap and Dunn's") logic is usually taken without this singularity restriction, allowing entailment to be a relation between *sets* of formulas. In this respect, the claim that Belnap and Dunn's four-valued logic and Anderson and Belnap's logic E_{fde} of first-degree entailment are "the same" would need additional explication.

The authors in this part elaborated deductive apparatus allowed for a further development towards some paraconsistent logics based on trilattices, and finally a generalized paraconsistent logic as a Gentzen-type sequent calculus with classical and paraconsistent negations. As to the trilattice-based logics, the authors present a sequent calculi L_{16} (propositional case) and F_{16} (first-order case) related to the trilattice $SIXTEEN_3$, which is a generalization of the algebraic structures suitable for Belnap-Dunn's logic. These calculi are shown to be cut-free, sound and complete with respect to the corresponding semantics. This construction is generalized further by formulating the cut-free, sound and complete Gentzen-style sequent calculi L_ω and F_ω comprising both classical and paraconsistent negations, which can be regarded as a kind of infinite-valued logics.

Part II presents an inquiry into a family of propositional logics with implications and co-implications extended with negations of various (in particular, paraconsistent) kinds. The most prominent among the logical systems, where an implication connective is intended to be counterbalanced by its dual, is the so-called bi-intuitionistic logic, which is now gaining more and more attention. This part contains an interesting discussion of the notion of constructivity in the context of bi-intutionistic logic and some of its extensions. It is argued that adding Nelson's paraconsistent strong negation to this logic opens different possibilities of defining a negated implication and co-implication, such as classical, intuitionistic, strongly constructive and connexive. It is shown how a cut-free and complete display calculus for Heyting-Brower logic in the style of Goré can be extended by the rules for constructively negated formulas. This logic can be equipped with a correct inferentialist semantics in terms of proofs, disproofs and their duals by supplementing the well-known Brower–Heyting–Kolmogorov interpretation by interpretations in terms of canonical disproofs, canonical reductions to absurdity (alias non-truth) and canonical reductions to non-falsity. Two other systems are considered in this part, namely symmetric paraconsistent logic and dual paraconsistent logic.

By studying this part, a reader may ponder an interesting question about an interrelation between the connectives of implication ($\to$) and co-implication

($\prec$) considered in the book. The authors adopt a widespread treatment of the latter as representing an algebraic operation of "pseudo-difference" (or "subtraction"), governed by the following residuation principle with respect to disjunction: $\gamma \leq \alpha \vee \beta \Leftrightarrow \gamma \prec \beta \leq \alpha$ (p. 135). In the literature this operation is often treated as the connective *dual* to implication. It is noteworthy that this treatment is *not* articulated in the book under review, and perhaps, this is nothing if not on purpose. As Peter Schroeder-Heister observes in his paper "Schluß und Umkehrschluß: ein Beitrag zur Definitionstheorie", in: C.F. Gethmann (ed.), *Deutsches Jahrbuch Philosophie 02. Lebenswelt und Wissenschaft*, Felix Meiner Verlag, Hamburg, 2009, pp. 1084–1085, such a treatment may be confusing. Indeed, if we compare the standard Kripke-style truth-conditions for implication and co-implication as presented on p. 142 of the book:

$\mathcal{M}, w \vDash^+ (\alpha \rightarrow \beta)$ iff for all $w' \geq w$: $\mathcal{M}, w \nvDash^+ \alpha$ or $\mathcal{M}, w \vDash^+ \beta$

$\mathcal{M}, w \vDash^+ (\alpha \prec \beta)$ iff there exists $w' \leq w$: $\mathcal{M}, w \vDash^+ \alpha$ and $\mathcal{M}, w \nvDash^+ \beta$,

one may notice that the so understood co-implication is in fact *not* the connective dual to implication, but its *converse*. A direct dualization of the truth-condition for $\rightarrow$ by interchanging between dual notions ("for all" and "there is", $\geq$ and $\leq$, *or* and *and*) produces a similar, but yet slightly different clause:

$\mathcal{M}, w \vDash^+ (\alpha \succ\!\!- \beta)$ iff there exists $w' \leq w$: $\mathcal{M}, w \nvDash^+ \alpha$ and $\mathcal{M}, w \vDash^+ \beta$,

where $\succ\!\!-$ stands for the dual implication. Analogously, a direct dualization of the introduction rules for implication in the display sequent calculi for the logics (I_i, C_i) on p. 149 (by interchanging between dual notions $\vdash$ and $\dashv$, $\circ$ and $\bullet$) would give us the following rules for the dual implication:

$$\frac{X \vdash \beta \quad \alpha \vdash Y}{X \bullet Y \vdash \alpha \succ\!\!- \beta} \ (\vdash\succ\!\!-) \qquad \frac{\alpha \bullet \beta \vdash X}{\alpha \succ\!\!- \beta \vdash X} \ (\succ\!\!-\vdash).$$

Part III explores an effect of employing paraconsistent connectives in the field of temporal logic. The primary focus is on a linear-time temporal logic LTL, which finds fruitful applications in automata-based model checking, specifying concurrent systems, etc. The authors formulate two bounded versions of LTL, which are extensions of constructive logics, and are intended to provide a useful theoretical basis for representing not only temporal, but also constructive and paraconsistent reasoning. In fact, it is a pioneering semantical and proof theoretical study of bounded constructive linear-time temporal logics involving intuitionistic and strong negations. Moreover, it is shown how an unbounded linear-time temporal logic can be obtained by adding to LTL a paraconsistent negation similar to the strong negation of N4. This logic is defined both semantically and proof-theoretically by suitable sequent and display calculi.

The last part of the book reflects a perfect suitability of a paraconsistent methodology to the area of substructural logics. In particular, logical systems of the following types are shown to have not only a pure logical import, but also some interesting applications in different branches: paraconsistent classical substructural logics, paraconsistent intuitionistic linear logics, paraconsistent logics based on involutive quantales, and paraconsistent Lambek logics. Remarkably, in all these logics the strong negation of Nelson's logic plays crucial role. Thus, an extended intuitionistic logic with strong negation can be regarded as a resource-conscious refinement of N4. In its turn, there is a direct correspondence between some fragment of this logic and the class of Petri nets with inhibitor arcs. This opens possibility of considering some realistic applications of the logics with strong negation not only in the theory of Petri nets with inhibitor arcs, but also in the theory of inconsistency-tolerant medical diagnosis, the theory of electrical circuits, and the theory of involutive quantales. In a similar vein, a kind of strong negation can be introduced into Categorial Grammar to form negated types, and to turn the functor-type forming directional implications of Categorial Grammar into connexive implications.

To sum up, the book under review is a clear demonstration of the extensive proof-theoretic power of the strong paraconsistent negation, which is at the heart of a paraconsistent version of Nelson's logic of constructible falsity. The authors manage to provide a convincing insight into the broad areas of the modern non-classical logic, connected in one way or other to this significant system, comprising both a paraconsistent and constructive idea. The book is lucidly organized, clearly written, and will be useful to both specialists in the field, as well as to a more general logical audience interested in the latest advances in proof theory.

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