Dual-Belnap Logic and Anything but Falsehood

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Abstract

This paper presents an inquiry into a proof system for a logic based on four Belnapian truth values, in which any truth value but the pure falsehood is designated. To this effect, I first implement a certain dualization of what Font terms 'Belnap's logic', and then show how it can be suitably extended. The resulting systems are of the Fmla-Set type dually to the standard formulation of Belnap's logic and the Exactly True Logic by Pietz and Rivieccio. I restate some philosophical motivation for the entailment relation of the Fmla-Set type by briefly comparing it with the usual Set-Fmla logical systems.

1 Preliminaries: Dunn and Belnap's four-valued semantics and designated truth values

J.Michael Dunn in his doctoral dissertation [9] initiated a strategy of semantic analysis, according to which sentences can systematically be considered not just true, or just false, but also neither true nor false, or both true and false simultaneously. This strategy has been technically implemented by constructing an 'intuitive semantics for first-degree entailment' in [10]; see also a comprehensive discussion (and generalization) of the subject in [11, 12]. Motivations for this approach may be various such as argumentative discourse, contradictory or incomplete theoretical systems, and philosophical paradoxes.

Following this strategy, Nuel Belnap [6, 7] introduced some weighty considerations from the computing field, in which sources and databases are often far from perfect, which forces the computers to deal with unreliable or corrupt information. Therefore, one arrives at four (generalized) truth values, according to the information that is 'told' to a computer with respect to a given sentence: 'just told True', 'just told False', 'told neither True nor False', 'told both True and False' (where 'True' and 'False' are ordinary classical truth values). It is most common to label these generalized truth values T, F, N, and B, respectively.

Let sentential language $\mathcal{L}$ be defined as follows:

$$\varphi ::= p \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid {\sim}\varphi.$$

In line with the principles of semantic analysis sketched above, define valuation v as a map from the set of sentential variables to the *subsets* of the set of classical truth-values $\{t, f\}$. This valuation is extended to the whole language by the following conditions:

Definition 1.1.

(1) $t \in v(\varphi \wedge \psi) \Leftrightarrow t \in v(\varphi)$ and $t \in v(\psi)$,
$f \in v(\varphi \wedge \psi) \Leftrightarrow f \in v(\varphi)$ or $f \in v(\psi)$;

(2) $t \in v(\varphi \vee \psi) \Leftrightarrow t \in v(\varphi)$ or $t \in v(\psi)$,
$f \in v(\varphi \vee \psi) \Leftrightarrow f \in v(\varphi)$ and $f \in v(\psi)$;

(3) $t \in v({\sim}\varphi) \Leftrightarrow f \in v(\varphi)$,
$f \in v({\sim}\varphi) \Leftrightarrow t \in v(\varphi)$.

Belnapian four truth values (being ascribed to a sentence φ) are then explicated as follows:

$v(\varphi) = B$ (told both True and False) $\Leftrightarrow t \in v(\varphi)$ and $f \in v(\varphi)$,

$v(\varphi) = T$ (just told True) $\Leftrightarrow t \in v(\varphi)$ and $f \notin v(\varphi)$,

$v(\varphi) = F$ (just told False) $\Leftrightarrow t \notin v(\varphi)$ and $f \in v(\varphi)$,

$v(\varphi) = N$ (told neither True nor False) $\Leftrightarrow t \notin v(\varphi)$ and $f \notin v(\varphi)$.

One therefore obtains an elegant semantic construction built on the "four values and three connectives" system, which can further be employed to determine entailment relation as a tool for "evaluating inferences", and finally, to obtain "logic, that is, a canon of inference" [6, p. 15]. This is normally done by marking certain truth values as *designated*, and by defining entailment as a relation that *preserves* this designated status in the course of reasoning.

Which values should be taken as designated among the Belnapian B, T, F, and N? This question can be answered differently depending on the underlying philosophical intuitions and the goals of logical analysis. Clearly, the pure falsehood F should be disqualified from the very outset. Four options are possible for the remaining three.

One option is to pick out $\{T, B\}$ as the set of designated truth values, which is the mainstream choice for Dunn and Belnap's four-valued semantics. A truth value is considered then to be designated if and only if it contains t (the classical True), being thus *at least true*. This choice is founded on the idea that entailment relation "never leads us from told True to the absence of told True (preserves Truth)" [3, p. 519] and brings about the system of 'tautological entailments' of relevant logic, see [2, Chapter III].

Alternatively, one can follow a dual intuition that "implication, entailment, validity, etc. should have as much to do with falsity preservation as with truth preservation—it is just that the direction is reversed" [10, p. 165]. Here it is important that valid entailment "never leads us from the absence of told False to told False (preserves non-Falsity)" [3, p. 519]. From this perspective a truth value is considered non-designated if and only if it is *at least not false*, containing thus f (the classical False), and the corresponding set of designated truth values will be $\{T, N\}$.

In the framework of the four-valued semantics, the above two choices are equivalent in the sense that they determine one and the same entailment relation (when defined on the same sets of premises and conclusions), see [10, p. 165], [12, p. 10], [13, Proposition 2.3]. Still, formally we have here two different choices of two different sets of designated truth values.

Andreas Pietz[1] and Umberto Rivieccio in [21] investigated a logic based on Belnapian four truth values, but with only T as designated. They give an informal motivation for such an 'exactly true logic', ensuring thus "a consequence relation that preserves *truth-and-non-falsity*" [21, p. 128]. This logic validates certain principles that are not valid in the original Dunn-Belnap's semantics, but is still not collapsed into classical logic.

One remaining option deserves attention: to allow as designated *any* truth value, *except the worst one*. According to Belnap, "the worst thing to be told is that something you cling to is false, *simpliciter*" [3, p. 516]. So, T is the "best of all" [ibid], N and B still hold out a hope of a better outcome, and only F is irrecoverable. Hence, it is reasonable to pose a question using the logic with $\{N, T, B\}$ as the subset of distinguished elements among the four Belnapian truth values. Such a logic should ensure preservation of everything but the (outright) falsehood.

João Marcos in [20] differentiates between entailment relations based on the sets of designated truth values $\{T\}$, $\{T, B\}$ and $\{N, T, B\}$. He shows how these relations

[1] After the name change—Andreas Kapsner.

can be explicated by "uniform classic-like semantical and proof-theoretical frameworks" in terms of bivaluations and the corresponding two-signed tableau systems. It is observed that "the inner structure of the four-valued formalism could be seen as a result from a natural combination of classical logic with itself" [20, p. 290].

The present paper is a companion article to [24] extending it by a detail examination of some characteristic features of the entailment relation based on the set of designated truth values $\{N, T, B\}$, and addresing the problem of its deductive formalization. Section 2 recalls a specific proof-theoretic characterization of the Dunn-Belnap semantics, considered by Josep Maria Font in [13] under the name 'Belnap's logic' in the form of a 'Hilbert-style calculus'. Section 3 briefly reviews a way of extending Belnap's logic, proposed by Pietz and Rivieccio to get their system ETL for a four-valued logic with T as the sole designated truth-value. Following this, Section 4 describes a dualization of Belnap's logic obtained by inverting its inference rules, and the corresponding definition of the entailment relation. Section 5 proceeds to certain extension of dual Belnap's logic resulting in logical system NFL ('non-falsity logic') for grasping the entailment relation for a backward preservation of the pure falsity (F). This system is proved to be sound and complete with respect to the intended semantics, and thus presents a solution of the stated problem. The paper is concluded with some philosophical explanations of the logics under consideration.

2 A Hilbert-style presentation of Belnap's logic

Font in [13, p. 5] associates 'Belnap's four-valued logic' with an entailment relation of the Set-Fmla[2] type, that is, "a relation $\models_{\mathcal{B}}$ between arbitrary sets of sentences and a sentence". Consider the following definition of $\models_{\mathcal{B}}$:

Definition 2.1. *Let* Γ *be any set of formulas, and* ψ *be any formula. Then* $\Gamma \models_{\mathcal{B}} \psi =_{df} \forall v : (\forall \varphi \in \Gamma : t \in v(\varphi)) \Rightarrow t \in v(\psi)$.[3]

This definition implies an acceptance of $\{T, B\}$ as the set of designated truth values, stating explicitly the preservation of classical truth (t) from premises to the conclusion. Moreover, the following lemma ensures the preservation of classical falsity (f) in a backward direction:

Lemma 2.2. $\Gamma \models_{\mathcal{B}} \psi \Leftrightarrow \forall v : f \in v(\psi) \Rightarrow (\exists \varphi \in \Gamma : f \in v(\varphi))$.

Proof. See proof of Lemma 2.2 in [24]. $\square$

[2]Cf. the classification of logical frameworks in [18, p. 198].

[3]Generally, Γ may be infinite, but in view of the well-known compactness property it is enough to consider some finite subset of Γ, cf. Definition 2.1 in [13].

This well-known result reinforces the point that Belnap's logic could equivalently be defined through the set of designated truth values $\{T, N\}$. Because both ways of characterizing $\models_{\mathcal{B}}$ are equivalent, we take the set $\{T, B\}$ determined by Definition 2.1 to be the canonical set of designated truth values for Belnap's logic.

For a proof-theoretic characterization of $\models_{\mathcal{B}}$, Font considers a specific system, which he describes as a "Hilbert-style axiomatization" of Belnap's logic [13, p. 10], denoting it $\vdash_H$. This system comprises only the so-called direct rules of inferences of the form $\Gamma \vdash \psi$ (organized vertically in a two-level shape), and has no axioms. The set of rules for $\vdash_H$ is as follows:

(R1) $\dfrac{\varphi \wedge \psi}{\varphi}$ (R2) $\dfrac{\varphi \wedge \psi}{\psi}$ (R3) $\dfrac{\varphi, \psi}{\varphi \wedge \psi}$

(R4) $\dfrac{\varphi}{\varphi \vee \psi}$ (R5) $\dfrac{\varphi \vee \psi}{\psi \vee \varphi}$ (R6) $\dfrac{\varphi \vee \varphi}{\varphi}$

(R7) $\dfrac{\varphi \vee (\psi \vee \chi)}{(\varphi \vee \psi) \vee \chi}$ (R8) $\dfrac{\varphi \vee (\psi \wedge \chi)}{(\varphi \vee \psi) \wedge (\varphi \vee \chi)}$ (R9) $\dfrac{(\varphi \vee \psi) \wedge (\varphi \vee \chi)}{\varphi \vee (\psi \wedge \chi)}$

(R10) $\dfrac{\varphi \vee \psi}{\sim\sim\varphi \vee \psi}$ (R11) $\dfrac{\sim\sim\varphi \vee \psi}{\varphi \vee \psi}$ (R12) $\dfrac{\sim(\varphi \vee \psi) \vee \chi}{(\sim\varphi \wedge \sim\psi) \vee \chi}$

(R13) $\dfrac{(\sim\varphi \wedge \sim\psi) \vee \chi}{\sim(\varphi \vee \psi) \vee \chi}$ (R14) $\dfrac{\sim(\varphi \wedge \psi) \vee \chi}{(\sim\varphi \vee \sim\psi) \vee \chi}$ (R15) $\dfrac{(\sim\varphi \vee \sim\psi) \vee \chi}{\sim(\varphi \wedge \psi) \vee \chi}$

Let us take a closer look at some deductive features of $\vdash_H$. Remarkably, it has no theorems (which is no surprise—no axioms, no theorems). Thus, this system is designed to establish (non-degenerate) valid consequences of the form $\Gamma \vdash \psi$, where Γ is non-empty. Elements of Γ can be called *assumption formulas*, and ψ is a *conclusion* derivable from Γ. Accounting for Font's characterization of $\vdash_H$ as a "Hilbert-style presentation", an inference (or derivation) of ψ from Γ in $\vdash_H$ should be defined as a finite *consecutive* list of (occurrences of) formulas, each of which either belongs to Γ or comes by an inference rule from some formulas preceding it in the list, and the last formula of which is ψ (cf. [19, p. 35]). If there is an inference of ψ from Γ in $\vdash_H$, then ψ is derivable from Γ in $\vdash_H$, and consequence $\Gamma \vdash \psi$ is said to be *valid* in $\vdash_H$.

Let $\Gamma \vdash_H \psi$ means that consequence $\Gamma \vdash \psi$ is valid in $\vdash_H$. By way of illustration, consider inferences for the following consequences: (a) $\varphi \vdash_H \sim\sim\varphi$ and (b) $\varphi \wedge \psi \vdash_H \sim\sim\varphi \wedge \psi$.

(a):

1. φ	(assumption)
2. $\varphi \vee {\sim\sim}\varphi$	1: (R4)
3. ${\sim\sim}\varphi \vee {\sim\sim}\varphi$	2: (R10)
4. ${\sim\sim}\varphi$	3: (R6)

(b):

1. $\varphi \wedge \psi$	(assumption)
2. φ	1: (R1)
3. ${\sim\sim}\varphi$	2: (a)
4. $\varphi \wedge \psi$	(assumption)
5. ψ	4: (R2)
6. ${\sim\sim}\varphi \wedge \psi$	3, 5: (R3)

For more examples of this inferential technique in systems like $\vdash_H$ one may wish to consult [15, pp. 125-126]. Observe, that taken literary (b) presents an inference $\varphi \wedge \psi, \varphi \wedge \psi \vdash_H {\sim\sim}\varphi \wedge \psi$. However, since Γ is considered to be a genuine set, consequence with a *contracted* assumptions set holds with no additional structural adjustments.

Interestingly Font, despite of his "Hilbert-style" characterization of $\vdash_H$, suggests also a construction of its inferences in a tree-like form resembling natural deduction, see [13, p. 11]. This suggestion can be exemplified by the following inferences of the consequences (a) and (b) above:

$$\text{(a)}\quad \dfrac{\dfrac{\dfrac{\varphi}{\varphi \vee {\sim\sim}\varphi}\,(R4)}{{\sim\sim}\varphi \vee {\sim\sim}\varphi}\,(R10)}{{\sim\sim}\varphi}\,(R6) \qquad\qquad \text{(b)}\quad \dfrac{\dfrac{\dfrac{\varphi \wedge \psi}{\varphi}\,(R1)}{{\sim\sim}\varphi}\,(a) \qquad \dfrac{\varphi \wedge \psi}{\psi}\,(R2)}{{\sim\sim}\varphi \wedge \psi}\,(R3)$$

As one can see, through such a construction inferences are evolving as direct derivations in the form of trees, possibly branching upwards. The derived formula constitutes the root of a tree, whereas its leaves stand for the formulas from which the root is derived (assumptions). Such form of inferences could be rather convenient and illustrative for explaining the main point of a proof-theoretic dualization of Belnap's logic, considered in Section 4 below.

The following fact helps to simplify inferences in $\vdash_H$ by eliminating extraneous disjunctions and turning disjunctions into conjunctions if required:

Lemma 2.3. *For every rule (R10)–(R15) of the form* $\dfrac{\varphi \vee \chi}{\psi \vee \chi}$ *the following rules are derivable in* $\vdash_H$*: (a)* $\dfrac{\varphi}{\psi}$*; (b)* $\dfrac{\varphi \wedge \chi}{\psi \wedge \chi}$.

Proof. See Proposition 3.2 in [13]. □

In particular, this lemma allows to establish all the properties of De Morgan negation for $\sim$. System $\vdash_H$ is sound and complete with respect to Definition 2.1:

Theorem 2.4. $\Gamma \vdash_H \psi \Leftrightarrow \Gamma \models_{\mathcal{B}} \psi$.

Proof. See Theorem 3.11 in [13]. □

3 Disjunction elimination and exactly true logic

As already observed, inference rules in $\vdash_H$ are all *direct* regulations ensuring a straightforward transition from premise(s) to conclusion. The first three rules deliver a complete inferential characterization of conjunction: (R1), (R2) for conjunction elimination and (R3) for conjunction introduction. The situation with disjunction is more intricate because the property of disjunction elimination is inexpressible within the Set-Fmla framework by a direct inference rule. Considering such an inexpressibility, this property is compensated in $\vdash_H$ by certain additional rules, most crucially, rules (R10)–(R15) with an additional disjunctive context attached to the usual double negation and De Morgan laws.

However, the property of disjunction elimination holds in $\vdash_H$ in a form of a *meta-principle* (or an admissible *meta-rule*), as is stated in the following lemma:

Lemma 3.1. *If $\varphi \vdash_H \chi$ and $\psi \vdash_H \chi$, then $\varphi \vee \psi \vdash_H \chi$.*

Proof. As observed in the proof of Proposition 3.3 in [13], if $\varphi \vdash_H \psi$, then $\varphi \vee \chi \vdash_H \psi \vee \chi$. Now, assume $\varphi \vdash_H \chi$ and $\psi \vdash_H \chi$. By the above observation, and using (R5), we obtain: (*) $\varphi \vee \psi \vdash_H \chi \vee \psi$ and (**) $\chi \vee \psi \vdash_H \chi \vee \chi$. The following inference completes the proof:

1. $\varphi \vee \psi$	(assumption)
2. $\chi \vee \psi$	1: (*)
3. $\chi \vee \chi$	2: (**)
4. χ	3: (R6)

□

Analogously, the property of contraposition is an admissible meta-principle in $\vdash_H$:

Lemma 3.2. *If $\varphi \vdash_H \psi$, then $\sim\psi \vdash_H \sim\varphi$.*

Proof. It is enough to show that the contrapositive versions of all the rules (R1)–(R15) are also the rules of $\vdash_H$. □

The absence of disjunction elimination among the derivable principles of $\vdash_H$ allows for some interesting extensions that would otherwise be impossible. For example, Pietz and Rivieccio [21] employ it to obtain a deductive characterization for their 'exactly true logic', which accepts T as the only designated truth value. Namely, consider the following definition:

Definition 3.3. $\Gamma \models_{\mathcal{T}} \psi =_{df} \forall v : (\forall \varphi \in \Gamma : v(\varphi) = T) \Rightarrow v(\psi) = T$.

The corresponding proof-system ETL can be obtained by extending $\vdash_H$ with the following rule of inference:[4]

(R16) $\dfrac{\varphi \wedge (\sim\varphi \vee \psi)}{\psi}$

Some properties of ETL are worthy of note. First, it validates *ex contradictione quodlibet*, that is, $\varphi \wedge \sim\varphi \vdash_{\mathrm{ETL}} \psi$ holds. However, contraposition and disjunction elimination are *not* admissible meta-principles of ETL; therefore, it does not collapse to classical or Kleene's logic. In particular, the classically valid consequence $\sim\psi \vdash \sim(\varphi \wedge \sim\varphi)$ (and more generally $\psi \vdash \varphi \vee \sim\varphi$) fails in ETL, which is evidence for the non-admissibility of contraposition. To see that disjunction elimination is also not admissible, it is sufficient to observe that $(\varphi \wedge \sim\varphi) \vee (\psi \wedge \sim\psi) \vdash \chi$ (valid in strong Kleene) fails in ETL, even though both $\varphi \wedge \sim\varphi \vdash_{\mathrm{ETL}} \chi$ and $\psi \wedge \sim\psi \vdash_{\mathrm{ETL}} \chi$ hold. Pietz and Rivieccio dub the latter property "anti-primeness" [21, p. 129].

ETL is sound and complete with respect to $\models_{\mathcal{T}}$:

Theorem 3.4. *$\Gamma \vdash_{\mathrm{ETL}} \psi \Leftrightarrow \Gamma \models_{\mathcal{T}} \psi$.*

Proof. See Theorem 3.4 in [21]. □

4 A dualization of Belnap's logic

Now, it is time to look at the four-valued consequence relations from a somewhat different (in fact, dual) perspective. Heinrich Wansing rightly remarks that "[t]he term 'duality' has several meanings even in mathematics" [27, p. 486]. However, as Michael Atiyah once noted, "[f]undamentally, duality gives *two different points of view of looking at the same object*" [4, p. 69].

Proceeding from the basic logical duality between Fregean *the True* and *the False* (to wit, classical t and f) one can first arrive at a duality between sentences of the object language $\mathcal{L}$ (with $\wedge$ and $\vee$), cf. [19, pp. 21-25], and then at a duality between expressions about consequence (most generally conceived as a relation between *arbitrary* sets of sentences of $\mathcal{L}$), based on the use of the concept of duality as "related to order reversal" [27, p. 486].

[4] In [21, p. 133] this rule is called "disjunctive syllogism", which is not quite accurate because the latter name is generally reserved for a slightly different principle, saying $\sim\varphi \wedge (\varphi \vee \psi) \vdash \psi$. In fact, (R16) presents an ordinary rule of *modus ponens* for a material conditional standardly defined through a disjunction in which the antecedent is negated, and in such a form this rule is often referred to in the literature as 'Ackermann's rule γ', see [1, p. 119].

Definition 4.1. *Let φ, ψ be any sentences of $\mathcal{L}$, and let φ^d be obtained from φ by interchanging between $\wedge$ and $\vee$, and replacing every atomic sentence with its negation (and likewise for ψ^d). Let $\Gamma^d = \{\varphi^d : \varphi \in \Gamma\}$, and $\Delta^d = \{\psi^d : \psi \in \Delta\}$ where Γ, Δ are non-empty sets of sentences of $\mathcal{L}$. Then $\Delta^d \vdash \Gamma^d$ is said to be* dual *to $\Gamma \vdash \Delta$.*

An easy induction on the length of a formula gives for any formula φ, and for any valuation v:

Lemma 4.2. $t \in v(\varphi) \Leftrightarrow f \in v(\varphi^d)$, *and* $f \in v(\varphi) \Leftrightarrow t \in v(\varphi^d)$.

Next step is to extend the notion of duality to logical systems (formulated in language $\mathcal{L}$) in general:

Definition 4.3. *Logical system L is said to be* self-dual *if $\Gamma \vdash_L \Delta \Leftrightarrow \Delta^d \vdash_L \Gamma^d$; logical systems L_1 and L_2 are said to be* mutually dual *if $\Gamma \vdash_{L_1} \Delta \Leftrightarrow \Delta^d \vdash_{L_2} \Gamma^d$.*

Notice again that Definitions 4.1 and 4.3 generally involve consequence expressions of the Set-Set framework. Expressions of Belnap's logic can be viewed as a special case of Set-Set consequence expressions with the singleton restriction in the succedent. Clearly, by definition, neither $\vdash_H$ nor ETL are self-dual, and cannot be such, precisely because they deal with the asymmetric consequence expressions of the Set-Fmla type.

This suggests a way of a *structural dualization* of $\vdash_H$ (and ETL) by constructing the corresponding logical system of a Fmla-Set framework. Namely, a 'Hilbert-style axiomatization' of the *dual Belnap logic* $\vdash_{dH}$ can be formulated as follows:

$$(\mathrm{R1}_d)\ \frac{\varphi}{\varphi \vee \psi} \qquad (\mathrm{R2}_d)\ \frac{\psi}{\varphi \vee \psi} \qquad (\mathrm{R3}_d)\ \frac{\varphi \vee \psi}{\varphi, \psi}$$

$$(\mathrm{R4}_d)\ \frac{\varphi \wedge \psi}{\varphi} \qquad (\mathrm{R5}_d)\ \frac{\varphi \wedge \psi}{\psi \wedge \varphi} \qquad (\mathrm{R6}_d)\ \frac{\varphi}{\varphi \wedge \varphi}$$

$$(\mathrm{R7}_d)\ \frac{(\varphi \wedge \psi) \wedge \chi}{\varphi \wedge (\psi \wedge \chi)} \qquad (\mathrm{R8}_d)\ \frac{(\varphi \wedge \psi) \vee (\varphi \wedge \chi)}{\varphi \wedge (\psi \vee \chi)} \qquad (\mathrm{R9}_d)\ \frac{\varphi \wedge (\psi \vee \chi)}{(\varphi \wedge \psi) \vee (\varphi \wedge \chi)}$$

$$(\mathrm{R10}_d)\ \frac{\sim\sim\varphi \wedge \psi}{\varphi \wedge \psi} \qquad (\mathrm{R11}_d)\ \frac{\varphi \wedge \psi}{\sim\sim\varphi \wedge \psi} \qquad (\mathrm{R12}_d)\ \frac{(\sim\varphi \vee \sim\psi) \wedge \chi}{\sim(\varphi \wedge \psi) \wedge \chi}$$

$$(\mathrm{R13}_d)\ \frac{\sim(\varphi \wedge \psi) \wedge \chi}{(\sim\varphi \vee \sim\psi) \wedge \chi} \qquad (\mathrm{R14}_d)\ \frac{(\sim\varphi \wedge \sim\psi) \wedge \chi}{\sim(\varphi \vee \psi) \wedge \chi} \qquad (\mathrm{R15}_d)\ \frac{\sim(\varphi \vee \psi) \wedge \chi}{(\sim\varphi \wedge \sim\psi) \wedge \chi}$$

This system manipulates consequence expressions of the Fmla-Set type, i.e., is designed to establish valid consequences of the form $\varphi \vdash \Delta$. Every inference in

$\vdash_{dH}$ has only one assumption, and a non-empty set of conclusions. Intuitively, an expression $\varphi \vdash \Delta$ means that at least one sentence among the elements of Δ is derivable from φ.[5]

To put it formally, an inference (or derivation) of Δ from φ in $\vdash_{dH}$ is a finite *consecutive* list of (occurrences of) formulas, the first of which is φ. All other formulas of the list are formed by applying the inference rules to formulas that precede these in the list, with Δ being the set of *terminating formulas* of the inference. A formula is terminating if and only if it has such an occurrence in the list, that is never used later as a premise of an inference rule applied in this inference. If there is an inference of Δ from φ in $\vdash_{dH}$, then Δ is derivable from φ in $\vdash_{dH}$, and consequence $\varphi \vdash \Delta$ is said to be *valid* in $\vdash_{dH}$.

By way of example consider the inferences in $\vdash_{dH}$ of the dual versions of formulas (a) and (b) above: (a_d) $\sim\sim\varphi \vdash_{dH} \varphi$ and (b_d) $\sim\sim\varphi \vee \psi \vdash_{dH} \varphi \vee \psi$.

(a_d):

1.	$\sim\sim\varphi$	(assumption)
2.	$\sim\sim\varphi \wedge \sim\sim\varphi$	1: ($\mathrm{R6}_d$)
3.	$\varphi \wedge \sim\sim\varphi$	2: ($\mathrm{R10}_d$)
4.	φ	3: ($\mathrm{R4}_d$), termination

(b_d):

1.	$\sim\sim\varphi \vee \psi$	(assumption)
2.	$\sim\sim\varphi$	1: ($\mathrm{R3}_d$)
3.	ψ	1: ($\mathrm{R3}_d$)
4.	$\varphi \vee \psi$	3: ($\mathrm{R2}_d$), term.
5.	φ	2: (a_d)
6.	$\varphi \vee \psi$	5: ($\mathrm{R1}_d$), term.

Note, that we had to infer the formula $\varphi \vee \psi$ twice, since without steps 5–6 the formula $\sim\sim\varphi$ would be terminating, and we would had the inference of $\sim\sim\varphi \vee \psi \vdash_{dH} \sim\sim\varphi, \varphi \vee \psi$ instead of (b_d).

One can also construct inferences in $\vdash_{dH}$ in a form of derivation trees, which—dually to the trees in $\vdash_H$—may branch downwards. Any derivation tree in $\vdash_{dH}$ has only one leaf, but can have many roots. The leaf of a tree stands for the formula from which its conclusions (roots) are derived. As an illustration consider the following derivation tree for (b_d):

$$\dfrac{\sim\sim\varphi \vee \psi}{\dfrac{\dfrac{\sim\sim\varphi}{\varphi}\,(\mathrm{a}_d)}{\varphi \vee \psi}\,(\mathrm{R1}_d) \qquad \dfrac{\psi}{\varphi \vee \psi}\,(\mathrm{R2}_d)}\,(\mathrm{R3}_d)$$

It is not difficult to obtain the dual version of Lemma 2.3:

[5]It is notable, that the reading of $\varphi \vdash \Delta$ as "φ entails ψ_1, or ψ_2, $\ldots$, or ψ_n, where $\psi_1, \psi_2, \ldots, \psi_n = \Delta$" is possible, but not inevitable. Rather the reading "φ entails ψ_i, for some $\varphi_i \in \Delta$" seems preferable.

Lemma 4.4. *For each rule* (Ri_d) *(*$10 \leqslant i \leqslant 15$*) of the form* $\dfrac{\varphi \wedge \chi}{\psi \wedge \chi}$*, the following rules hold: (a)* $\dfrac{\varphi}{\psi}$*, and (b)* $\dfrac{\varphi \vee \chi}{\psi \vee \chi}$.

The duality between $\vdash_H$ and $\vdash_{dH}$ are established by the following theorem:

Theorem 4.5. $\Gamma \vdash_H \psi \Leftrightarrow \psi^d \vdash_{dH} \Gamma^d$.

Proof. To prove this theorem it is enough to observe that the dual version of every rule of $\vdash_H$ is derivable in $\vdash_{dH}$ and vice versa. ☐

It is most natural to define entailment relation of the Fmla-Set type as a relation that backwardly preserves classical falsity (f) from conclusions to the assumption:

Definition 4.6. $\varphi \vDash_{\mathcal{DB}} \Delta =_{df} \forall v : (\forall \psi \in \Delta : f \in v(\psi)) \Rightarrow f \in v(\varphi)$.

One can observe a semantical duality between $\vDash_{\mathcal{B}}$ and $\vDash_{\mathcal{DB}}$:

Theorem 4.7. *For any* Γ*, for any* $\psi : \Gamma \vDash_{\mathcal{B}} \psi \Leftrightarrow \psi^d \vDash_{\mathcal{DB}} \Gamma^d$.

Proof. Consider arbitrary Γ and ψ. Let $\forall v : (\forall \varphi \in \Gamma : t \in v(\varphi)) \Rightarrow t \in v(\psi)$. Assume, $\exists v : (\forall \varphi^d \in \Gamma^d : f \in v(\varphi^d))$ and $f \notin v(\psi^d)$. By using Lemma 4.2 one very quickly gets a contradiction. The proof of the converse is similar. ☐

Definition 4.6 explicitly suggests $\{T, N\}$ as the set of designated truth values (and hence, dually to Definition 2.1, $\{F, B\}$ as the set of *non-designated* values). Still, just like $\vDash_{\mathcal{B}}$, relation $\vDash_{\mathcal{DB}}$ preserves classical truth in the forward direction, as the following lemma states, being obtained by a simple dualization of Lemma 2.2:

Lemma 4.8. $\varphi \vDash_{\mathcal{DB}} \Delta \Leftrightarrow \forall v : t \in v(\varphi) \Rightarrow (\exists \psi \in \Delta : t \in v(\psi))$.

Proof. For every valuation v define its dual v^*, such that $t \in v^*(p) \Leftrightarrow f \notin v(p)$, and $f \in v^*(p) \Leftrightarrow t \notin v(p)$. A direct induction extends this valuation to any formula of the language. Now, assume $\varphi \vDash_{\mathcal{DB}} \Delta$. Consider an arbitrary valuation v, such that $\forall \psi \in \Delta : t \notin v(\psi)$. We have then $\forall \psi \in \Delta : f \in v^*(\psi)$, and hence, $f \in v^*(\varphi)$. Thus, $t \notin v(\varphi)$. The proof of the converse is similar. ☐

Dually to $\vdash_H$, conjunction introduction and contraposition are inexpressible in $\vdash_{dH}$ as direct inference rules. Nevertheless, admissibility of the corresponding meta-principles can be obtained in $\vdash_{dH}$ by dualization of Lemma 3.1 and Lemma 3.2:

Lemma 4.9. *If* $\varphi \vdash_{dH} \psi$ *and* $\varphi \vdash_{dH} \chi$*, then* $\varphi \vdash_{dH} \psi \wedge \chi$.

Lemma 4.10. *If $\varphi \vdash_{dH} \psi$, then $\sim\psi \vdash_{dH} \sim\varphi$.*

System $\vdash_{dH}$ is sound with respect to Definition 4.6:

Theorem 4.11. *If $\varphi \vdash_{dH} \Delta$, then $\varphi \vDash_{\mathcal{DB}} \Delta$.*

Proof. For every rule (R1$_d$), (R2$_d$), and (R4$_d$)–(R15$_d$) of the form $\dfrac{\alpha}{\beta}$ assume $f \in v(\beta)$. Then an assumption that $f \notin v(\alpha)$ will lead to a contradiction. For (R3$_d$) assume $f \in v(\varphi)$ and $f \in v(\psi)$. Then we obtain a contradiction from the assumption that $f \notin v(\varphi \vee \psi)$. □

To obtain completeness of $\vdash_{dH}$ with respect to Definition 4.6 one can employ a technique of bringing any formula of $\mathcal{L}$ to a *normal form*. In what follows I dualize appropriately the definitions and proofs from [13, pp. 12-14]. Let *Var* be the set of propositional variables of $\mathcal{L}$, and $Lit = Var \cup \{\sim p : p \in Var\}$ be the set of *literals*. Let Cl be the set of *clauses*—the least set of formulas containing Lit and closed under $\wedge$. Let $var(\varphi)$ be the set of variables of φ, and $var(\Gamma)$ be the set of variables of formulas from Γ. For any clause φ the set of its literals $lit(\varphi)$ is defined inductively by: $lit(\varphi) = \{\varphi\}$ if $\varphi \in Lit$, and $lit(\varphi \wedge \psi) = lit(\varphi) \cup lit(\psi)$. For $\Gamma \subseteq Cl$, $lit(\Gamma)$ is the set of literals of formulas from Γ. As usual, $\varphi \dashv\vdash_{dH} \psi$ means $\varphi \vdash_{dH} \psi$ and $\psi \vdash_{dH} \varphi$, and likewise for $\vDash\!\vDash_{\mathcal{DB}}$.

Lemma 4.12. *For all $\varphi \in (\mathcal{L})$ there is a finite $\Gamma \subseteq Cl$, such that $var(\varphi) = var(\Gamma)$, and for any $\psi \in \mathcal{L}$, for all $\gamma \in \Gamma : \varphi \wedge \psi \dashv\vdash_{dH} \gamma \wedge \psi$.*

Proof. By induction on the length of φ. If $\varphi \in Var$, then we can put $\Gamma = \{\varphi\}$. Let $\varphi = \varphi_1 \wedge \varphi_2$, and Γ_1, Γ_2 correspond to φ_1, φ_2 by inductive hypothesis. Then $\Gamma = \{\gamma_1 \wedge \gamma_2 : \gamma_1 \in \Gamma_1, \gamma_2 \in \Gamma_2\}$ satisfies $var(\Gamma) = var(\varphi)$, and we have: $(\varphi_1 \wedge \varphi_2) \wedge \psi \dashv\vdash_{dH} \varphi_1 \wedge (\varphi_2 \wedge \psi)$ (by R7$_d$) $\dashv\vdash_{dH} \gamma_1 \wedge (\varphi_2 \wedge \psi)$ (for all $\gamma_1 \in \Gamma_1$, by inductive hypothesis) $\dashv\vdash_{dH} \varphi_2 \wedge (\gamma_1 \wedge \psi)$ (for all $\gamma_1 \in \Gamma_1$, by a principle derivable in $\vdash_{dH}$) $\dashv\vdash_{dH} \gamma_2 \wedge (\gamma_1 \wedge \psi)$ (for all $\gamma_1 \in \Gamma_1, \gamma_2 \in \Gamma$, by inductive hypothesis) $\vdash_{dH}$) $\dashv\vdash_{dH} (\gamma_2 \wedge \gamma_1) \wedge \psi)$ (for all $\gamma_1 \in \Gamma_1, \gamma_2 \in \Gamma$, by converse of R7$_d$ derivable in $\vdash_{dH}$). The cases with $\varphi = \varphi_1 \vee \varphi_2$ and $\varphi = \sim\varphi_1$ are analogous. □

Lemma 4.13. *For any $\varphi \in \mathcal{L}$ there is a finite $\Gamma \subseteq Cl$, such that $var(\varphi) = var(\Gamma)$ and $\varphi \dashv\vdash_{dH} \bigvee \Gamma$.*

Proof. Similarly as above, by induction on the length of φ. Let $\varphi \in Var$. Then $\Gamma = \{\varphi\}$. Let $\varphi = \varphi_1 \wedge \varphi_2$, and Γ_1, Γ_2 correspond to φ_1, φ_2 by inductive hypothesis. Then $\Gamma = \{\gamma_1 \wedge \gamma_2 : \gamma_1 \in \Gamma_1, \gamma_2 \in \Gamma_2\}$ satisfies $var(\Gamma) = var(\varphi)$, and we have: $\varphi \dashv\vdash_{dH} \gamma_1 \wedge \varphi_2$ (for all $\gamma_1 \in \Gamma_1$, by Lemma 4.12) $\dashv\vdash_{dH} \varphi_2 \wedge \gamma_1$ (for all $\gamma_1 \in \Gamma_1$, by R5$_d$)$\dashv\vdash_{dH}$ $\gamma_2 \wedge \gamma_1$ (for all $\gamma_1 \in \Gamma_1, \gamma_2 \in \Gamma_2$, by Lemma 4.12) $\dashv\vdash_{dH}$ $\gamma_1 \wedge \gamma_2$ (for all $\gamma_1 \in \Gamma_1, \gamma_2 \in \Gamma_2$, by R5$_d$). Hence, $\varphi \vdash_{dH} \bigvee\{\gamma_1 \wedge \gamma_2 : \gamma_1 \in \Gamma_1, \gamma_2 \in \Gamma_2\}$ (by R1$_d$). To get the converse, assume $\bigvee\{\gamma_1 \wedge \gamma_2 : \gamma_1 \in \Gamma_1, \gamma_2 \in \Gamma_2\}$. By R3$_d$, $\{\gamma_1 \wedge \gamma_2 : \gamma_1 \in \Gamma_1, \gamma_2 \in \Gamma_2\}$, and since φ is equivalent through $\dashv\vdash_{dH}$ to each $\gamma_1 \wedge \gamma_2$, we thus obtain φ. The cases with $\varphi = \varphi_1 \vee \varphi_2$ and $\varphi = {\sim}\varphi_1$ are analogous. □

Lemma 4.14. *Every $\varphi \in \mathcal{L}$ is equivalent both through $\dashv\vdash_{dH}$ and $\Dashv\vDash_{\mathcal{DB}}$ to a disjunction of clauses with the same variables.*

Proof. For $\dashv\vdash_{dH}$ the lemma holds by Lemma 4.13. Due to Theorem 4.11 it holds for $\Dashv\vDash_{\mathcal{DB}}$ as well. □

Lemma 4.15. *Let $\varphi \in Cl$, $\Delta \subseteq Cl$. Then $\varphi \vDash_{\mathcal{DB}} \Delta \Rightarrow \varphi \vdash_{dH} \Delta$.*

Proof. Assume $\varphi \vDash_{\mathcal{DB}} \Delta$. For a fixed $\varphi \in Cl$ define valuation v^l on literals by putting for every $p \in Var$: $t \in v^l(\varphi) \Leftrightarrow p \in lit(\varphi)$; $f \in v^l(\varphi) \Leftrightarrow {\sim}p \in lit(\varphi)$. By Definition 4.6, $(\forall\psi \in \Delta : {\sim}p \in lit(\psi)) \Rightarrow {\sim}p \in lit(\varphi)$, and by Theorem 4.8 $p \in lit(\varphi) \Rightarrow \exists\psi \in \Delta : p \in lit(\psi)$, for every p. Thus, $lit(\varphi) \subseteq lit(\psi)$. Since both φ and ψ are clauses, ψ is a conjunction of the same literals appearing in φ, and maybe other ones, modulo some associations, permutations, repetitions, etc. By using (R4$_d$), (R5$_d$), (R6$_d$) and (R7$_d$), we get $\varphi \vdash_{dH} \psi$, and thus, $\varphi \vdash_{dH} \Delta$. □

Theorem 4.16. *For any φ and Δ : if $\varphi \vDash_{\mathcal{DB}} \Delta$, then $\varphi \vdash_{dH} \Delta$.*

Proof. By Lemma 4.15 and Lemma 4.14. □

I finish this section with a brief review of some well-known notions and results from *abstract algebraic logic*, as displayed, e.g., in [13, 16, 17], adjusted to a Fmla-Set framework. Assume a standard notion of *logical matrix* for language $\mathcal{L}$ as a pair $\langle \mathbf{A}, D\rangle$, where $\mathbf{A}$ is an algebra of type $\mathcal{L}$ with universe A, and $D \subseteq A$. If A forms a *lattice*, then D is a *lattice filter* on A which can also be prime. The *Leibniz congruence* $\Omega_{\mathbf{A}}(D)$ of the matrix $\langle \mathbf{A}, D\rangle$ is defined as the largest congruence of $\mathbf{A}$, such that if any two elements $a, b \in \mathbf{A}$ are connected by the congruence relation and $a \in D$, then $b \in D$ as well. A matrix is said to be *reduced* if its Leibniz congruence is the identity relation.

Consider a structural consequence relation of a Fmla-Set type, i.e. a relation $\vdash \subseteq \mathcal{L} \times P(\mathcal{L})$ satisfying the following properties for all $\varphi, \psi \in \mathcal{L}$ and all $\Gamma, \Delta \subseteq \mathcal{L}$:

Reflexivity: $\varphi \vdash \{\varphi\} \cup \Delta$.
Monotonicity: if $\varphi \vdash \Gamma$, then $\varphi \vdash \Gamma \cup \Delta$.
Transitivity: if $\varphi \vdash \Gamma$ and $\psi \vdash \{\varphi\} \cup \Delta$, then $\psi \vdash \Gamma \cup \Delta$.
Structurality: if $\varphi \vdash \Delta$, then $\sigma\varphi \vdash \sigma\Delta$, for every uniform substitution σ on $\mathcal{L}$.

A logic L in a Fmla-Set framework can be then defined as a pair $\langle \mathcal{L}, \vdash \rangle$. A logical matrix is considered to be a *model* of a logic L when $\varphi \vdash_L \Delta$ implies for any valuation v on A (a homomorphism from $\mathcal{L}$ to $\mathbf{A}$) $v(\psi) \in D$ (for some $\psi \in \Delta$), whenever $v(\varphi) \in D$. In such a case the set D is called a filter for L or an L-filter. A logic L is said to be *complete* relative to a class of its matrix models iff for every $\Delta \cup \{\varphi\} \subseteq \mathcal{L}$, such that $\varphi \nvdash_L \Delta$, there is a logical matrix $\langle \mathbf{A}, D \rangle$ (which is a model of L) and a valuation $v \in Hom(\mathcal{L}, A)$, such that $v(\varphi) \in D$ but $v(\psi) \notin D$, for every $\psi \in \Delta$. It is well-known that every logic is complete with respect to the class of all its reduced models, see, e.g., [14, p. 207].

Observe, that the set of Belnapian truth values $\{T, B, N, F\}$ constitute a lattice with operations of meet, join and involution that correspond to the connectives determined by Definition 1.1. This lattice labeled in [13, p. 3] as $\mathfrak{M}_4$ generates the variety of *De Morgan lattices* **DM**. Famously, $\mathfrak{M}_4$ has exactly two prime filters $D_b = \{T, B\}$ and $D_n = \{T, N\}$.

Theorem 4.11 in fact demonstrates that matrix $\langle \mathfrak{M}_4, D_n \rangle$ is a model of $\vdash_{dH}$. By using Theorem 4.16, it can also be shown that $\vdash_{dH}$ is complete with respect to the class of logical matrices $\langle \mathbf{A}, D \rangle$, where $\mathbf{A}$ is **DM** and D is the set of filters generated by D_n (or equivalently by D_b). Note, that D is closed under intersections, being thus itself a complete lattice.

In the next section I will need the following lemma, which can be obtained from Theorem 3.14 in [13]:

Lemma 4.17. *If* $\mathbf{A}$ *is a non-trivial (i.e. not one-element) algebra, then* $\langle \mathbf{A}, D \rangle$ *is a reduced matrix for* $\vdash_{dH}$ *iff* $\mathbf{A} \in \mathbf{DM}$ *and* D *is a lattice filter of* $\mathbf{A}$.

5 The non-falsity logic and dual γ

Definition 4.6 explicates the entailment relation of the dual Belnap logic as essentially preserving classical falsity (f) in a *backward direction*. We can strengthen this property, and consider a relation that is backwardly hereditary with respect to Belnapian *exact falsity*:

Definition 5.1. $\varphi \vDash_{\mathcal{F}} \Delta =_{df} \forall v : (\forall \psi \in \Delta : v(\psi) = F) \Rightarrow v(\varphi) = F$.

Dually to Definition 3.3, this relation takes F as *the only non-designated* truth value, and thus, is based on the set of designated values $\{T, B, N\}$. As explained in [24, p. 1308], such a choice may be suitable if we wish "to allow as designated all the truth values *except the worst one*", and thus, to consider "anything but the (outright) falsehood".

The following theorem establishes semantical duality between $\vDash_{\mathcal{T}}$ and $\vDash_{\mathcal{F}}$:

Theorem 5.2. *For any* Γ*, for any* $\psi : \Gamma \models_{\mathcal{T}} \psi \Leftrightarrow \psi^d \models_{\mathcal{F}} \Gamma^d$.

Proof. Similarly as the proof of Theorem 4.7. $\square$

This duality suggests a deductive formalization of the non-falsity logic (NFL) on the basis of the dual Belnap logic obtained by extending system $\vdash_{dH}$ by the *dual Ackermann's rule* γ:[6]

(R16$_d$) $\dfrac{\varphi}{\psi \vee ({\sim}\psi \wedge \varphi)}$

Theorem 5.3. $\Gamma \vdash_{\mathrm{ETL}} \psi \Leftrightarrow \psi^d \vdash_{\mathrm{NFL}} \Gamma^d$.

Proof. In addition to the proof of Theorem 4.5 one has to consider (R16) and to state that its dual version is derivable in NFL, and analogously with derivability of the dual version of (R16$_d$) in ETL. $\square$

NFL is sound with respect to Definition 5.1:

Theorem 5.4. *If* $\varphi \vdash_{\mathrm{NFL}} \Delta$*, then* $\varphi \models_{\mathcal{F}} \Delta$.

Proof. A simple check confirms the fact that every rule (R1$_d$)–(R16$_d$) preserves the truth value F from conclusions to the premise. $\square$

NFL is a paraconsistent system, since $\varphi \wedge {\sim}\varphi \vdash \psi$ is not NFL-derivable. Indeed, assume $v(\psi) = F$, and take $v(\varphi) = B$. Then $v(\varphi \wedge {\sim}\varphi) = B$, and hence, $\varphi \wedge {\sim}\varphi \not\models_{\mathcal{F}} \psi$. By Theorem 5.4 $\varphi \wedge {\sim}\varphi \nvdash_{\mathrm{NFL}} \psi$.

But NFL is not paracomplete as the following derivation shows:

1. φ	(assumption)
2. $\psi \vee ({\sim}\psi \wedge \varphi)$	1: (R16$_d$)
3. ψ	2: (R3$_d$)
4. ${\sim}\psi \wedge \varphi$	2: (R3$_d$)
5. ${\sim}\psi$	4: (R4$_d$)
6. $\psi \vee {\sim}\psi$	3: (R1$_d$), termination
7. $\psi \vee {\sim}\psi$	5: (R2$_d$), termination

[6]In [24] we used the rule of *dual disjunctive syllogism* in the form $\frac{\varphi}{{\sim}\psi \vee (\psi \wedge \varphi)}$.

Moreover, the principle of conjunction introduction is not admissible in NFL. Indeed, we have both $\varphi \vdash_{\mathrm{NFL}} \psi \vee {\sim}\psi$ and $\varphi \vdash_{\mathrm{NFL}} \chi \vee {\sim}\chi$. But if we take $v(\varphi) = T$, $v(\psi) = B$ and $v(\psi) = N$, we obtain $v((\psi \vee {\sim}\psi) \wedge (\chi \vee {\sim}\chi)) = F$, thus $\varphi \not\models_{\mathcal{F}} (\psi \vee {\sim}\psi) \wedge (\chi \vee {\sim}\chi)$, and hence, $\varphi \nvdash_{\mathrm{NFL}} (\psi \vee {\sim}\psi) \wedge (\chi \vee {\sim}\chi)$.

To prove the completeness of NFL with respect to Definition 5.1 one can dualize an algebraic technique employed in [21]. Consider again lattice $\mathfrak{M}_4$ with the lattice order $\leqslant$. Let $x \preceq y$ generally stands for the lattice equation $x \sqcap y \approx x$. Since $\mathfrak{M}_4$ generates the variety of De Morgan lattices **DM**, it satisfies an equation $x \approx y$ iff this equation is satisfied in all the lattices from **DM**. Once $\langle \mathcal{L}, \wedge, \vee, {\sim} \rangle$ is known to form a De Morgan lattice, we have:

Lemma 5.5. *For every $\varphi, \psi_1, \ldots, \psi_n \in \mathcal{L}$ the following are equivalent:*

(i) $\varphi \models_{\mathcal{F}} \psi_1, \ldots, \psi_n$;

(ii) $\mathfrak{M}_4$ *satisfies* ${\sim}(\psi_1 \vee \ldots \vee \psi_n) \wedge \varphi \preceq \psi_1 \vee \ldots \vee \psi_n$.

Proof. $(i) \Rightarrow (ii)$: Assume (i), and consider an arbitrary valuation v. If $v(\psi_1 \vee \ldots \vee \psi_n) = T$, the lemma holds. If $v(\psi_1 \vee \ldots \vee \psi_n) = N$, then $v({\sim}(\psi_1 \vee \ldots \vee \psi_n)) = N$, and $N \sqcap x \leqslant N$ holds for any $x \in \{T, F, B, N\}$. The same argument holds for $v(\psi_1 \vee \ldots \vee \psi_n) = B$. If $v(\psi_1 \vee \ldots \vee \psi_n) = F$, then by (i) $v(\varphi) = F$, and (ii) holds as well.

$(ii) \Rightarrow (i)$: Assume (ii), and consider a valuation v, such that $v(\psi_1 \vee \ldots \vee \psi_n) = F$. Then $T \wedge v(\varphi) \leqslant F$, and thus, $v(\varphi) = F$. □

Combining this lemma with the algebraic implications of the completeness result for $\vdash_{dH}$ from the previous section, we get the desired theorem:

Theorem 5.6. *If $\varphi \models_{\mathcal{F}} \Delta$, then $\varphi \vdash_{\mathrm{NFL}} \Delta$.*

Proof. Assume $\varphi \nvdash_{\mathrm{NFL}} \Delta$. This implies that there is some reduced matrix $\langle \mathbf{A}, D \rangle$, and a valuation $v \in Hom(\mathcal{L}, A)$, such that $v(\varphi) \in D$ but $v(\psi) \notin D$, for every $\psi \in \Delta$. Since NFL is an extension of $\vdash_{dH}$, this matrix will also be a model of $\vdash_{dH}$, and by Lemma 4.17, $\mathbf{A} \in \mathbf{DM}$ and D is a lattice filter of $\mathbf{A}$. Note, that D is closed under (R16$_d$). Now, suppose $\varphi \models_{\mathcal{F}} \Delta$. By Lemma 5.5, $\langle \mathbf{A}, D \rangle$ satisfies ${\sim}(\psi_1 \vee \ldots \vee \psi_n) \wedge \varphi \preceq \psi_1 \vee \ldots \vee \psi_n$, were $\psi_1, \ldots, \psi_n = \Delta$. Hence, $v({\sim}(\psi_1 \vee \ldots \vee \psi_n) \wedge \varphi) \leqslant v(\psi_1 \vee \ldots \vee \psi_n)$, and thus, $v({\sim}(\psi_1 \vee \ldots \vee \psi_n)) \wedge \varphi)) \notin D$. But since D is closed under (R16$_d$), we should have $v({\sim}(\psi_1 \vee \ldots \vee \psi_n)) \wedge \varphi)) \in D$, a contradiction. □

6 Concluding remarks: feasibility of Fmla-Set entailment

This paper elaborates a general method of dualizing the proof systems of certain kind, which can be referred to as 'degenerated' Hilbert-style axiomatic systems, which have in fact no axioms, but only direct inference rules of the Set-Fmla type. Syntactically the dualization in question consists just in reversing all the inference rules of the system to be dualized (together with the proper dualization of all the involved sentences), and in switching thus to a system that deals now with Fmla-Set consequences. The semantic definition of the entailment relation is subject to the analogous dualization, which reflects a general duality between truth and falsity.

The described procedure was performed on two systems—Font's formulation of Belnap's logic and Pietz and Rivieccio's formulation of exactly true logic, both based on a four-valued Belnapian semantics, but with different choices of designated truth values. As a result we obtain two new systems, formalizing the dual Belnap logic and the non-falsity logic, which belong to the Fmla-Set framework. Soundness and completeness of these systems with respect to the corresponding four-valued semantics were established.

In view of the technical considerations of the present paper a reader may not feel comfortable with the very idea of a logic formulated in the Fmla-Set framework. If we agree that "logic is the science of argument" [23, p. ix], what kind of argument could comprise logical systems of such type?

It may be noted that Belnap's logic, and specifically system $\vdash_H$ defined in a Set-Fmla framework, is an exemplar of what can be called *a logic of proof*, where an argument is conceived as a procedure of *proving some sentence* by inferring it from a collection of premises. In such a setting an argument is just a logical device that ultimately "leads to a conclusion, *one* conclusion, or so one would think" [25, p. 333]. In this sense, as observed in [23, p. ix], "ordinary arguments are lopsided: they can have any number of premisses but only one conclusion". But is any kind of logical deduction necessarily such?

Shoesmith and Smiley in their now classic book [23], drawing on pioneering insights by Gerhard Gentzen, Rudolf Carnap and William Kneale, advance a *multiple-conclusion logic* that allows "any number of conclusions as well, regarding them ... as setting out the field within which the truth must lie if the premisses are to be accepted" [23, p. ix]. Since then, the subject of a multiple-conclusion logic has been taken up by various authors, see [22] for a prominent example of this.

However, Florian Steinberger recently challenged the very idea of multiple conclusions by appealing to standards of logical inferentialism, "the position that the meanings of the logical constants are determined by the rules of inference they obey" [25, p. 333]. This position, he argues, is incompatible with multiple-conclusion proof systems because such systems are supposedly not "connected to our ordinary deductive inferential practices", and thus, he says, "constitute a departure from our ordinary forms of inference and argument" [25, pp. 335, 340].

Even leaving a dubious issue of finding logical structures "in nature" (see [25, pp. 339]) aside, one can point out serious limitations of an inferentialist conception based on the notion of proof (to wit, *assertion*) only. Wansing, for instance, pays particular attention to the speech act of *denial* "in the context of a use-based, inferentialist account of linguistic meaning" [27, p. 483], distinguishing then between provability, disprovability, and their duals, where "the dual of provability is reducibility to non-truth", and "the dual of disprovability is reducibility to non-falsity" [27, p. 486]. He also considers the corresponding 'inferential relations'[7], and moreover, supplements the well-known *Brouwer-Heyting-Kolmogorov (BHK) interpretation* of the logical (intuitionistic) connectives formulated in terms of canonical proofs by "interpretations in terms of canonical disproofs, canonical reductions to absurdity (alias non-truth), and canonical reductions to non-falsity" [27, p. 493].

Luca Tranchini by constructing a natural deduction system for dual-intuitionistic logic observes a close correspondence between introduction rules of the natural deduction system for intuitionistic logic and Brouwer-Heyting-Kolmogorov 'proof-interpretation' of the logical constants. He suggests a dual 'refutation-interpretation' for the logical constants through the 'dual-BHK' clauses, which in turn correspond to elimination rules of the natural deduction system for dual-intuitionistic logic, see [26, p. 645-646]. Note, incidentally, that the natural deduction system for dual-intuitionistic logic constructed by Tranchini is "a single-premise multiple-conclusions system in which derivation trees branch downward" [26, p. 632].

The latter observation not only supports the justifiability of multiple-conclusion systems, but also highlights the relevance of an entailment relation *with only one premise*, and particularly the systems considered in Sections 4 and 5 above. Namely, dual Belnap logic, as well as non-falsity logic, belonging to the Fmla-Set logical framework, can be most naturally considered a kind of what Kosta Došen once called *logics of refutation*:

> A refutation would be a deduction where we have at most one premise; from this premise we try to deduce a number of conclusions, with the intent to show that all these conclusions are refutable, so that the premise

[7]In particular, the consequence of the form $\varphi \vdash \psi_1, \ldots, \psi_n$ stands for an inferential relation "φ is reducible to absurdity from *counterassumptions* $\psi_1, \ldots, \psi_n$".

must be refutable too. Sequents are read backwards: if all sentences on the right are refutable, a sentence on the left is refutable. [8, p. 111]

In this way, systems $\vdash_{dH}$ and NFL restore an essential lopsidedness of the 'ordinary proof-arguments', but in a dual fashion, as the pure 'refutation-arguments', which can have any number of conclusions but only one premise. The set of conclusions forms then a *refutation set* for a given premise, whereas the premise stands for a hypothesis to be tested for refutability.

Systems $\vdash_{dH}$ and NFL differ in their selection criteria to possible refutations. A sentence can generally be considered refutable if it can take a non-designated truth value. Then, in dual Belnap logic, it is sufficient for a sentence to be not (classically) true, *or* to be at least (classically) false, depending on the chosen set of designated truth values (either $\{T, B\}$, or $\{T, N\}$—both options are possible on an equal footing), to qualify as a refutation in a given inference. By contrast, in the non-falsity logic, the criterion is much stronger—here a genuine refutation must be both false *and* not true.

Both these systems, being interesting in their own right, indicate a general usefulness of the Fmla-Set logical framework for certain logico-methodological purposes, and thus, its worthiness for further elaboration.

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Received 18 February 2018