THE DIAMOND OF MINGLE LOGICS: A FOUR-FOLD INFINITE WAY TO BE SAFE FROM PARADOX

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ABSTRACT. System **R**-Mingle (**RM**) was invented by J. Michael Dunn in the middle of the 1960s. This system got its name due to the characteristic logical principle called "Mingle." Although this principle allows for certain irrelevant inferences, it can protect us from (the worst effects of) the paradoxes of relevance. Furthermore, separating the first-degree entailment fragment of a mingle logic allows one to concentrate on the characteristic principle of that fragment, known as "Safety." Based on a purely Tarskian formulation of first-degree entailment systems, four types of Safety can be distinguished and corresponding proof systems can be constructed, forming a diamond-shaped lattice with infinitely many systems between its vertices. The corner systems of the diamond can be supplied with uniform and rather natural semantics, which reaffirms the rightful place of the mingle logics in the family of the first-degree entailment systems.

Keywords. Binary consequence system, First-degree entailment, Generalized truth-value functions, Paradoxes of relevance, R-Mingle, Variable-sharing property

1. INTRODUCTION

The first-degree entailment fragment of Dunn's logic **R**-Mingle (**RM**) has occasionally appeared in the literature under various names and characterizations. For one, Makinson in [43, p. 38] presents it as a *system of Kalman implication*, reflecting the fact that the algebraic counterpart of its characteristic axiom $x \cap -x \leq y \cup -y$ was first considered by Kalman [42] for defining a "normal" lattice with involution (*i-lattice* for short, nowadays standardly called *De Morgan lattice*). Dunn in [32, p. 43] also pays tribute to Kalman by calling *Kalman consequence system* essentially the same system, but formulated with a turnstile instead of an arrow. Dunn also notes that it is in fact the first-degree entailment fragment of the relevance logic **RM**. In Ermolaeva and Muchnik [37] still the same system is presented as a "fragment of Łukasiewicz's logic," and Dunn in [34, p. 15] observes as well that the system, which comprises "the first-degree entailments of Dunn and McCall's '**R**-Mingle' [. . .] is also the first-degree entailment fragment of Łukasiewicz's 3-valued logic."

There is also a tradition of naming Kalman's normal i-lattice *Kleene algebra*. Apparently, this tradition was initiated by Brignole and Monteiro, see [22, p. 4, especially Definition 2.4], and then continued by Cignoli [23], Balbes and Dwinger [9], Blyth

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and Varlet [21], and others. However, this word usage can be misleading in two ways. First, as Dunn remarks [35, p. 444], it should not be confused with Kleene algebras "which arise in the study of relation algebras and regular expressions" (see, e.g., Bimbó and Dunn [18] and also [19, Ch. 7]). Second, confusion may arise by extrapolating this terminology to a logical level, which happens in Font [38, p. 26], where it was proposed to define "Kleene's three-valued logic" through $\varphi \wedge {\sim}\varphi \vdash \psi \vee {\sim}\psi$ as the characteristic consequence. This proposal is unfortunate indeed, taking into account the fact that for Kleene's logic a more general principle $\varphi \wedge {\sim}\varphi \vdash \psi$ is usually considered to be characteristic.

Rivieccio in [54, p. 325] tries to reconcile the above mentioned algebraic tradition with logical usage by introducing the name "Kleene's logic of order" for "the logic that corresponds to the lattice order of Kleene lattices" in the sense of Brignole and Monteiro. (He also corrects a mistake in Font's deductive formalization of this logic based on his "Hilbert-style system.") It is unclear how much this new term will help to clear up the confusion. In Section 7, I will explain why it is better to associate the system in question with a specific fragment of **R**-Mingle or Łukasiewicz's logic rather than with that of Kleene's logic.

Anyway, one can only agree with the assessment of Albuquerque, Přenosil and Rivieccio [1, p. 1025] that this logic "has received considerably less attention in the literature" than some of its cousins, such as Kleene's strong three-valued logic and Priest's Logic of Paradox. This lack of attention appears to be unjustified given certain important features of the mingle principle, particularly, its ability to neutralize so-called *paradoxes of relevance*.

In this paper, I will explain, in which sense this principle can secure us against paradoxes even if they appear in our logic, which thus turns out to be "semi-relevant" (Section 2). Moreover, since the separation of the first-degree entailment fragment of a mingle logic makes it possible to leave out further irrelevant properties (such as the "Chain Property"), I will focus on the characteristic principle of that fragment, known as "Safety" (Section 3). Based on a "purely Tarskian" formulation of the first-degree entailment systems (Section 4), I will differentiate between four types of Safety and construct the corresponding proof systems (Section 5). It turns out that these systems form a diamond-shaped lattice with an infinite number of systems connecting its vertices. Furthermore, the diamond's corner systems can be supplied with uniform and rather natural semantics (Section 6), reaffirming the mingle logics' rightful place in the family of first-degree entailment systems.

An important caveat. When in this paper I speak of "first-degree entailment" I always mean a relation between *single formulas*, as it is originally conceived by Belnap, see [11]. Occasionally, I also involve the entailment relation between *sets* of formulas and *formulas*, but only for the sake of comparison.

2. Preliminaries. R-Mingle and Variable Sharing Property

J. Michael Dunn's contributions to the evolution of modern logic are significant and varied. His achievements in investigating and eliminating so-called *paradoxes of relevance*, in particular, are widely acknowledged. These usually refer to certain properties of material (and strict) implication, which may be true even if its antecedent and consequent have nothing to do with each other (are mutually irrelevant). Central among these paradoxes are:

(Positive Paradox) $$\varphi \to (\psi \to \varphi),$$

which says that a true proposition is implied by any proposition whatsoever, and

(Negative Paradox) $$\sim\varphi \to (\varphi \to \psi),$$

according to which a false proposition is implied by any proposition whatever it might be.[1]

Objections to these paradoxes have given rise to a whole branch of logical investigation, *relevance logic*, initiated by pioneering work of Wilhelm Ackermann, Alan Ross Anderson and Nuel Belnap, see [3].[2] People who have chosen relevance logic as a field of their scientific interest — Mike Dunn among them — are often called *relevance logicians*.

One can describe a relevance logician as a person who "seeks an entailment connective $\to$ which is such that $A \to B$ holds only if B is relevant to A" Copeland [24, p. 325]. In search of such a connective Belnap [12] has proposed a certain criterion that is now considered a necessary condition for any logic ($\mathfrak{L}$) to be relevant, the so-called *variable sharing property* (VSP):

$\varphi \to \psi$ is a theorem of $\mathfrak{L}$ only if φ and ψ share a sentential variable.

As is well known, all of the major relevance logic systems, such as **B**, **T**, **R** and **E**, have this property. However, in the vicinity of relevance logics, there is a remarkable system that lacks VSP. Dunn invented this system, known as **R**-Mingle (**RM**), in the mid-1960s. The system gets its name from a corresponding logical principle, that is characteristic for it. The principle in question has its origin in a paper by Ohnishi and Matsumoto [48], where the term "mingle" was used for the following rule:

(Mingle rule) $$\frac{\Gamma \vdash \Theta \quad \Sigma \vdash \Pi}{\Gamma, \Sigma \vdash \Theta, \Pi}$$

Anderson and Belnap in [3, p. 97] consider a version of mingle in the context of an intuitionistic sequent system, dealing with sequents having at most one formula in the succedents:

(A–B Mingle) $$\frac{\Gamma \vdash \varphi \quad \Sigma \vdash \varphi}{\Gamma, \Sigma \vdash \varphi}.$$

[1] Strictly speaking, the formulas just presented are most commonly called "paradoxes of material implication." I will use a more general name, however, considering these formulas as paradigmatic representatives of a wider group of implicative statements (and also consequence expressions), in which "the antecedents and consequents (or premises and conclusions) are on completely different topics" (Mares [44]), or *irrelevant* to each other. Note also, that the terms "Positive Paradox" and "Negative Paradox" are sometimes used differently by different authors. In this paper, I will use these terms in the sense just described, namely, to specify the situations in which truth (maybe necessary, or logical) is implied (or entailed) by any proposition, and falsehood (maybe necessary, or logical) implies (or entails) any proposition.

[2] Another powerful branch of modern logic, which arose from a consideration of these paradoxes is, of course, modal logic.

They observe, that adding this rule to a system with the following "Arrow on the right" rule:

$$(\vdash\!\to) \qquad \frac{\Gamma, \varphi \vdash \psi}{\Gamma \vdash \varphi \to \psi},$$

yields the following derived rule (even in the absence of Weakening, and provided that $\vdash \varphi \to \psi$ and $\varphi \vdash \psi$ are interderivable, see Dunn [33, p. 149]):

$$\text{(O–M Mingle)} \qquad \frac{\varphi \to \gamma \quad \psi \to \gamma}{\varphi \to (\psi \to \gamma)}.$$

The latter leads to the following axiom with the same name (in the presence of $\varphi \to \varphi$):

$$\text{(Mingle axiom)} \qquad \varphi \to (\varphi \to \varphi).$$[3]

Dunn in [33, p. 146] tells an interesting story about inventing **R**-Mingle, formulated by him in a Storrs McCall's graduate seminar at the University of Pittsburgh (cf. [3, p. 94]). By modifying a suggestion of McCall, Dunn simply added Mingle to the set of axioms of relevance logic **R**, and formulated thus the system **RM**, the first appearance of which in print seems to be Dunn [27]. In that paper Dunn shows that, based on Meyer's completeness result for **RM**, every proper normal extension of **RM** has a finite characteristic matrix, despite the fact that **RM** lacks such a matrix.[4] As a result, **RM** is pretabular (or has the so-called *Scroggs's property*, see [33, p. 147]). Furthermore, Dunn demonstrates the strong completeness of **RM** with respect to Sugihara matrices.

Despite the perception (perhaps not quite unjustified) that "**RM** deserves more respect than it has gotten" [33, p. 142], it should be noted that many important aspects of **R**-Mingle and some of its fragments were discussed in detail by different authors, Avron [5; 6; 7; 8], Blok and Raftery [20], Meyer [46], Metcalfe [45], Parks [50], Robles, Méndez and Salto [55] among them. Moreover, it is sometimes argued that the "Dunn–McCall logic **RM** is by far the best understood and the most well-behaved logic in the family of logics developed by the school of Anderson and Belnap" [8, p. 15].

As to the variable sharing property, although **RM** does not possess it in full generality, see [3, p. 417], it still has the following *weak variable sharing property* (WVSP):

> If $\varphi \to \psi$ is a theorem of **RM**, then either (i) φ and ψ share a sentential variable, or (ii) both $\sim\varphi$ and ψ are theorems of **RM**. (Cf. [3, p. 417] and [33, p. 142].)

As Meyer has shown (see [27, p. 4]), the following formula is provable in **RM**.

$$\text{(ii)} \qquad \sim(\varphi \to \varphi) \to (\psi \to \psi)$$

This theorem is denoted by (ii) here because it exemplifies exactly item (ii) in the definition of WVSP by representing an implication in which the negation of a theorem implies (even if irrelevantly) a theorem. It is worthy of note that Mingle and (ii) are

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[3] My presentation of various "mingles" (either in the form of a rule or an axiom) employs a unified symbolism. "A–B Mingle" and "O–M Mingle" mark the contributions of Anderson and Belnap, and Ohnishi and Matsumoto, respectively. The latter label is taken from [33, p. 152], where it stands for a corresponding axiom. In what follows when I say "Mingle," I will always mean the Mingle *axiom*.

[4] An extension of **RM** is called *normal* iff it is closed under substitution and the **RM**-rules of inference.

interderivable within **RM**, see Schechter [57], so $\mathbf{R}+$(ii) yields **RM**. Furthermore, there is another remarkable property of **RM**, that appears to be related to the paradoxes of relevance and states, in effect, that a false proposition implies *any* true proposition. Dunn calls this principle *Ex Falso Verum*:

$$\text{(EFV)} \qquad {\sim}\varphi \to (\psi \to (\varphi \to \psi)).$$

Mingle is provable in $\mathbf{R}+$EFV (see [33, pp. 154–155]), and thus, all three principles — Mingle, (ii) and EFV — turn out to be equivalent within **RM**. However, unlike Mingle, principles (ii) and EFV allow to identify explicitly a subtle but rather important distinction between (1) being *free* from the paradoxes of relevance, and (2) being *safe* from them. This distinction between freedom and safety deserves special consideration. The first, of course, implies the second, but there may well be the second without the first. That is to say, there may well be irrelevant inferences which, nevertheless, are guaranteed to do no harm, and occurrence of which does not cause damage to our knowledge.

Imagine we are developing some theory, and we do so in a standard way by establishing certain axioms (which are by definition true), and then continue to prove theorems, step by step. We want to avoid at least two bad situations. *First*, we want to rule out the possibility of inferring irrelevant conclusions from *true* premises, i.e., we want our arguments for what follows from the axioms to be always on point. *Second*, even if a false statement is inadvertently introduced into our theory, we want to avoid the multiplication of falsity, i.e., we want falsehood to remain isolated and, at the very least, not reproduce itself (till we will be able to find and remove it). As soon as these minimal conditions are met, we feel safe and unaffected by contradictions, even if some of them manage to penetrate our theory.

It turns out that **RM** is quite capable to guarantee the fulfillment of those two conditions. Indeed, according to condition (ii) of WVSP and its proof-theoretic counterpart, even if our inference comes to be irrelevant, the worst thing that can happen, is that we stay with our truths. While the variable sharing property is meant to ensure freedom from paradoxes of relevance, its weak version (WVSP) is well suited to prevent their possible destructive effect.

It is exactly because of WVSP that **RM** is often regarded as a "semi-relevant" system, see, e.g., [3, p. 375], [65, p. 768]. Recently Avron [8] elaborated a more precise notion of "semi-relevance" which encompasses logics in which (1) for every two sets of formulas Γ, Δ, and any formula ψ we have $\Gamma \vdash \psi$ whenever $\Gamma \cup \Delta \vdash \psi$ and Δ has no atomic formulas in common with $\Gamma \cup \{\psi\}$; and which (2) does not have a finite weakly characteristic matrix. Avron proved that **RM** is semi-relevant in this sense.

Dunn in [33, p. 157] specifies some useful properties of **RM**, which make it desirable for use as a "logical tool." Namely, it is decidable, has a low complexity, and can be equipped with a simple, easy-to-understand Kripke-style semantics with a binary accessibility relation, which can be extended to obtain a constant domain semantics for quantifiers. Although **RM** does not have the variable sharing property, it is semi-relevant in the sense of Avron, and it is paraconsistent "in the sense that contradiction does not imply every sentence whatsoever ('Explosion')" [33, p. 160]. Dunn observes that whereas **RM** does have some irrelevant implications like (ii), they are "safe in that, unlike Explosion lead to nothing new" [33, p. 161].[5]

In this context another principle derivable in **RM** might be even more telling, expressing *explicitly* the property of being safe from the paradoxes of relevance as discussed above; namely,

(Safety) $$(\varphi \wedge \sim\varphi) \to (\psi \vee \sim\psi).$$

Safety essentially says the same thing as (ii), but with a single implication as the main connective, which opens up the possibility of explaining the idea of protection against paradoxes of relevance on the level of *first-degree entailment* (by replacing implication with a consequence relation). It is called "Safety" in [34, p. 14], because, as Dunn observes, with this principle we can always feel safe: "even if a theory has a contradiction as a theorem, all that can be derived from it are tautologies" [35, p. 443]. Note, that although an informal content of Safety is similar to (ii), their deductive strength is not the same. The former is still weaker than the latter, since neither Mingle, nor (ii) is derivable in **R** + Safety, whereas Safety is derivable not only in **RM**, but also in **R** + (ii), see [57].

The significance of Safety may become clearer if we consider another **RM** property located between Mingle and Safety, which is rather counter-intuitive from a "relevance standpoint" — the so-called *Chain Property* expressed by the formula

(CP) $$(\varphi \to \psi) \vee (\psi \to \varphi).$$

As Dunn explains, "[i]t says that given two possibly very distinct sentences, say p = 'The moon is made of green cheese,' and q = 'The cat is on the mat,' one of the two will imply the other" [33, p. 161]. It is hardly possible to find for CP any kind of intuitive justification similar to the one available for (ii) and Safety, and thus, from a relevantist perspective, the validity of the Chain Property may be taken as a "serious weakness for **RM**" [ibid.].

The claim that CP is located "between" (ii) and Safety has an exact sense, since these three principles constitute a non-reversible "chain of derivabilities." (1) CP is derivable in **R** + (ii), but (ii) is not derivable in **R** + CP; and (2) Safety is derivable in **R** + CP, but CP is not derivable in **R** + Safety (see [57, p. 120], taking also into account that Mingle and (ii) are interderivable as mentioned above). As a result, if we want to remove the aforementioned weakness from **RM**, focusing on Safety as "the most safe" result of adding Mingle to **R** may be promising. To that end, one can begin by separating a specific fragment of **RM** that retains some of its most useful features, but lacks the Chain Property. This is the *first-degree entailment* fragment of **RM**.

3. First-degree Entailment and Safety from the Paradoxes

First-degree entailment is a rather remarkable field of study, which has arisen within relevance logic research program, and to which Dunn's contribution is not only

[5] Dunn provides a rather witty illustration of how an irrelevant inference can be harmless within **RM**, which I quote here in full: "I have been trying to think of an analogy and the best I have been able to come up with goes something like this. Suppose I am building an electrical circuit and I want to protect against faults. Normally, a small fault will turn all the switches on. But what if I somehow insert a clever circuit that allows a switch to be turned on only if it is already on?" [33, p. 158].

significant, but indeed fundamental. The idea of first-degree entailment has been introduced by Belnap already in his doctoral dissertation [10] and then put into print in [11], who considered an expression of the form $\varphi \to \psi$ a first-degree entailment iff "φ and ψ are both written solely in terms of propositional variables, $\wedge$, $\vee$, and $\sim$ (other truth-functional connectives being treated as defined by these)" [11, notation adjusted]. Thus, by considering first-degree entailments $\varphi \to \psi$, "where φ and ψ can be truth functions of any degree but cannot contain any arrows," one "ignores the possibility and problems of *nested entailments*" [3, p. 150, italics mine].

To grasp this idea, keep in mind that Belnap originally conceived of first-degree entailment as a fragment of the relevance logic system **E** (of entailment). The latter, being the favorite system of Anderson and Belnap, has been designed to provide a "formal analysis of the notion of logical implication, variously referred to also as 'entailment,' [...] expressed in such logical locutions as 'if ... then–,' 'implies,' 'entails,' etc., and answering to such conclusion-signaling logical phrases as 'therefore,' 'it follows that,' 'hence,' 'consequently,' and the like" [3, p. 5]. According to this interpretation, the system **E** represents an object-language theory that explains the properties of the entailment relation by identifying it (within this theory) with an object-language implicational connective. In this way, nested implications express statements about entailments between entailments; for instance, $\varphi \to (\psi \to \chi)$ says that φ entails that ψ entails χ.

However, if we want to treat entailment as a meta-language relation in its own right (i.e., "as signifying a metalinguistic relation of logical consequence" [3, p. 150]), it may be appropriate to separate "logical implication" from the rest of the propositional connectives, and consider expressions, in which statements formed only with these remaining connectives are *consequences* of the others. To make this separation more explicit, one can use turnstile ($\vdash$) instead of arrow ($\to$), and obtain in this way *consequence expressions*. Among these expressions one can distinguish the so-called *binary consequence expressions* (cf. [32, p. 24]), or expressions from the FMLA-FMLA logical framework, see [39, p. 198], which represent consequences between *single* formulas. (Dunn and Hardegree consider in this respect "binary implicational systems" [36, p. 194].) A *binary consequence system* is then a proof system which manipulates consequence expressions as formal objects. Anderson and Belnap's concept of first-degree entailment corresponds to the concept of a binary consequence system. Let me summarize (and in a way generalize) this by means of precise definitions.

If $\{\circ_1, \dots, \circ_m\}$ is a set of binary, and $\{\diamond_1, \dots, \diamond_n\}$ — a set of of unary propositional connectives, then propositional language $\mathcal{L}_{\{\circ_1,\dots,\circ_m,\diamond_1,\dots,\diamond_n\}}$ can be defined as usual:

$$\varphi ::= p \mid \varphi \circ_1 \varphi \mid \dots \mid \varphi \circ_m \varphi \mid \diamond_1 \varphi \mid \dots \mid \diamond_n \varphi.$$

Definition 1. A *binary consequence relation* $\vdash$ over $\mathcal{L}_{\{\circ_1,\dots,\circ_m,\diamond_1,\dots,\diamond_n\}}$ is a subset of the set $\mathcal{L}_{\{\circ_1,\dots,\circ_m,\diamond_1,\dots,\diamond_n\}} \times \mathcal{L}_{\{\circ_1,\dots,\circ_m,\diamond_1,\dots,\diamond_n\}}$.

Definition 2. A *binary consequence expression* (or simply a consequence) of $\mathcal{L}_{\{\circ_1,\dots,\circ_m,\diamond_1,\dots,\diamond_n\}}$ is a pair $(\varphi, \psi) \in \vdash$, usually written as $\varphi \vdash \psi$ (to be read as "φ has ψ as a consequence" Dunn [31, p. 302]), where $\varphi, \psi \in \mathcal{L}_{\{\circ_1,\dots,\circ_m,\diamond_1,\dots,\diamond_n\}}$.

Definition 3. A *logic* in language $\mathcal{L}_{\{\circ_1,\dots,\circ_m,\diamond_1,\dots,\diamond_n\}}$ is a binary consequence relation over $\mathcal{L}_{\{\circ_1,\dots,\circ_m,\diamond_1,\dots,\diamond_n\}}$, closed at least under the usual Tarskian conditions of *reflexivity* and *transitivity*:

$$\text{(ref)} \quad \varphi \vdash \varphi, \qquad\qquad \text{(tr)} \quad \varphi \vdash \psi,\ \psi \vdash \chi \Rightarrow \varphi \vdash \chi.$$[6]

Definition 4. A *binary consequence system* $(\mathcal{L}_{\{\circ_1,\ldots,\circ_m,\diamond_1,\ldots,\diamond_n\}}, \vdash)$ in the language $\mathcal{L}_{\{\circ_1,\ldots,\circ_m,\diamond_1,\ldots,\diamond_n\}}$ is a proof system, which manipulates binary consequences of $\mathcal{L}_{\{\circ_1,\ldots,\circ_m,\diamond_1,\ldots,\diamond_n\}}$ as formal objects.

Definition 5. Let $\mathfrak{L}$ be a logical system formulated in language $\mathcal{L}_{\{\circ_1,\ldots,\circ_m,\rightarrow,\diamond_1,\ldots,\diamond_n\}}$ with an implication $\rightarrow$ among its binary connectives. Then binary consequence system $\mathfrak{L}_{\mathsf{fde}} = (\mathcal{L}_{\{\circ_1,\ldots,\circ_m,\diamond_1,\ldots,\diamond_n\}}, \vdash)$ is the *first-degree entailment fragment* of $\mathfrak{L}$ iff for any $\varphi, \psi \in \mathcal{L}_{\{\circ_1,\ldots,\circ_m,\diamond_1,\ldots,\diamond_n\}}$, $\vdash_{\mathfrak{L}} \varphi \rightarrow \psi \Leftrightarrow \varphi \vdash_{\mathfrak{L}_{\mathsf{fde}}} \psi$.

In Section 4, I will present a more precise definition of the notion of a binary consequence system (see Definition 12). For now, Definition 4 will suffice for our purposes.

I now return to system **E** and Belnap's elaboration of the idea of first-degree entailment in the context of this system. **E** is usually formulated in the language $\mathcal{L}_{\{\rightarrow,\wedge,\vee,\sim\}}$. In his doctoral dissertation [10] (see also [13]) Belnap introduced a proof system in the same language, which was then presented in [3, §5.2] under the label $\mathbf{E}_{\mathsf{fde}}$. A distinctive feature of that system is that one of the connectives, namely, implication $(\rightarrow)$ could have only one occurrence in a formula of the system as the main operator. I reproduce here this system under the same label as a binary consequence system formulated in language $\mathcal{L}_{\{\wedge,\vee,\sim\}}$.

System $\mathbf{E}_{\mathsf{fde}}$

(ce_1)	$\varphi \wedge \psi \vdash \varphi$	(ce_2)	$\varphi \wedge \psi \vdash \psi$
(di_1)	$\varphi \vdash \varphi \vee \psi$	(di_2)	$\psi \vdash \varphi \vee \psi$
(ni)	$\varphi \vdash \sim\sim\varphi$	(ne)	$\sim\sim\varphi \vdash \varphi$
(dis_1)	$\varphi \wedge (\psi \vee \chi) \vdash (\varphi \wedge \psi) \vee \chi$	(tr)	$\varphi \vdash \psi,\ \psi \vdash \chi \,/\, \varphi \vdash \chi$
(ci)	$\varphi \vdash \psi,\ \varphi \vdash \chi \,/\, \varphi \vdash \psi \wedge \chi$	(de)	$\varphi \vdash \chi,\ \psi \vdash \chi \,/\, \varphi \vee \psi \vdash \chi$
(con)	$\varphi \vdash \psi \,/\, \sim\psi \vdash \sim\varphi$		

The following theorem shows that $\mathbf{E}_{\mathsf{fde}}$ is indeed the first-degree entailment fragment of system **E**.

Theorem 6. *For any $\varphi, \psi \in \mathcal{L}_{\{\wedge,\vee,\sim\}}$, $\varphi \rightarrow \psi$ is provable in* **E** *iff $\varphi \vdash \psi$ is provable in* $\mathbf{E}_{\mathsf{fde}}$.

Proof. Following Belnap [11], define *primitive entailment* as a consequence $\chi_1 \wedge \cdots \wedge \chi_k \vdash \xi_1 \vee \cdots \vee \xi_l$, in which every χ_i and ξ_j is an *atom* (i.e., a propositional variable or its negation). A primitive entailment is *explicitly tautological* iff some atom χ_i is the same as some ξ_j. Now, a consequence $\varphi \vdash \psi$ represents a *tautological entailment* iff it is reducible by replacements through commutativity, associativity, distributivity, De Morgan and double negation rules to a consequence $\varphi_1 \vee \cdots \vee \varphi_m \vdash \psi_1 \wedge \cdots \wedge \psi_n$, where every $\varphi_i \vdash \psi_j$ is an explicitly tautological entailment.

It can be shown that a formula $\varphi \rightarrow \psi$ (where $\varphi, \psi \in \mathcal{L}_{\{\wedge,\vee,\sim\}}$) is provable in **E** iff $\varphi \vdash \psi$ represents a tautological entailment, see [2, p. 14]. Moreover, it is

[6] Normally, Tarskian conditions for a consequence relation include monotonicity ($\varphi \vdash \psi \Rightarrow \Gamma, \varphi \vdash \psi$), but it is inexpressible in the FMLA-FMLA framework.

known that $\varphi \vdash \psi$ represents a tautological entailment iff it is provable in $\mathbf{E}_{\mathsf{fde}}$, see [3, pp. 159–161]. $\lhd$

As Dunn observes, already in his doctoral dissertation, this proof may be easily adapted to demonstrate that $\mathbf{E}_{\mathsf{fde}}$ is also the first-degree entailment fragment of system **R**, "which shows that **E** and **R** agree in their first degree entailment fragments" [25, p. 115]. To highlight this fact, Dunn in [30, p. 146] calls this system $\mathbf{R}_{\mathsf{fde}}$. In [34] a somewhat different formulation of the system labeled as $\mathbf{R}_{\mathsf{fde}}$ is presented, in which the contraposition rule (con) is replaced by four De Morgan laws, which are derivable in $\mathbf{E}_{\mathsf{fde}}$.

$$(dm_1)\quad {\sim}(\varphi \vee \psi) \vdash {\sim}\varphi \wedge {\sim}\psi \qquad\qquad (dm_2)\quad {\sim}\varphi \wedge {\sim}\psi \vdash {\sim}(\varphi \vee \psi)$$
$$(dm_3)\quad {\sim}(\varphi \wedge \psi) \vdash {\sim}\varphi \vee {\sim}\psi \qquad\qquad (dm_4)\quad {\sim}\varphi \vee {\sim}\psi \vdash {\sim}(\varphi \wedge \psi)$$

It is not difficult to demonstrate that (con) remains admissible (although not derivable) in the system so formulated, see [34, Proposition 11]. In what follows, I will mark by $\mathbf{R}_{\mathsf{fde}}$ the system with De Morgan laws taken as axioms instead of contraposition rule (con), while retaining the label $\mathbf{E}_{\mathsf{fde}}$ for the original formulation from [3] with the contraposition rule. In view of the admissibility of (con) in $\mathbf{R}_{\mathsf{fde}}$ and derivability of (dm_1)–(dm_4) in $\mathbf{E}_{\mathsf{fde}}$, both formulations are deductively equivalent in the sense that they determine the same set of provable consequences.

We now turn to Dunn's fundamental contribution to the metatheory of first-degree entailment. In his doctoral dissertation [25] (see also the seminal paper [28]), Dunn provides $\mathbf{E}_{\mathsf{fde}}$ with an intuitively appealing semantics, the main point of which is to allow for underdetermined and overdetermined valuations, allowing a sentence to be *rationally* considered to be *neither* true nor false, as well as *both* true and false. This has given rise to a highly innovative research program in modeling entailment, which is sometimes called "the American Plan," see Routley [56], (cf. also Shramko [59]). Belnap in [14; 15] has developed Dunn's idea further by introducing specific truth values for such non-standard valuations. These new truth values allow for a quite natural informational interpretation, namely, as information that has been communicated, say, to a computer.

Let $\{t, f\}$ be the set of classical truth values. Define a *generalized truth value function* v^4 as a map from the set of propositional variables into the *subsets* of $\{t, f\}$. These subsets can then be considered *generalized truth values* understood as "mere truth" ($T = \{t\}$), "mere falsehood" ($F = \{f\}$), "neither truth nor falsehood" ($N = \{\ \}$), and "both truth and falsehood" ($B = \{t, f\}$). Function v^4 is then determined on the set $\{T, F, N, B\}$, and can be extended to compound formulas by the following definition.

Definition 7.

(1) $t \in v^4(\varphi \wedge \psi) \;\Leftrightarrow\; t \in v^4(\varphi)$ and $t \in v^4(\psi)$,
$f \in v^4(\varphi \wedge \psi) \;\Leftrightarrow\; f \in v^4(\varphi)$ or $f \in v^4(\psi)$;
(2) $t \in v^4(\varphi \vee \psi) \;\Leftrightarrow\; t \in v^4(\varphi)$ or $t \in v^4(\psi)$,
$f \in v^4(\varphi \vee \psi) \;\Leftrightarrow\; f \in v^4(\varphi)$ and $f \in v^4(\psi)$;
(3) $t \in v^4({\sim}\varphi) \;\Leftrightarrow\; f \in v^4(\varphi)$,
$f \in v^4({\sim}\varphi) \;\Leftrightarrow\; t \in v^4(\varphi)$.

Generalized truth values can be explicated as the outcomes of applying the function v^4 to propositions of our language, being thus the elements from the power-set of the set of classical truth values, $\mathcal{P}(\{t,f\})$:

$$\begin{array}{lcll} v^4(\varphi)=T & \Leftrightarrow & t\in v^4(\varphi) \text{ and } f\notin v^4(\varphi) & (\varphi \text{ is true only});\\ v^4(\varphi)=F & \Leftrightarrow & t\notin v^4(\varphi) \text{ and } f\in v^4(\varphi) & (\varphi \text{ is false only});\\ v^4(\varphi)=B & \Leftrightarrow & t\in v^4(\varphi) \text{ and } f\in v^4(\varphi) & (\varphi \text{ is both true and false});\\ v^4(\varphi)=N & \Leftrightarrow & t\notin v^4(\varphi) \text{ and } f\notin v^4(\varphi) & (\varphi \text{ is neither true nor false}). \end{array}$$

Note the difference between expressions $v^4(\varphi)=T$ and $t\in v^4(\varphi)$. Whereas the former expression says that φ is *only* true (i.e., true and not false), the latter means that φ is *at least* true (which does not exclude that it can be false as well). And similarly for the expressions $v^4(\varphi)=F$ and $f\in v^4(\varphi)$.

We have the following definition of entailment as a relation between single formulas (of the FMLA-FMLA type).

Definition 8. $\varphi \vDash_{\mathsf{fde}} \psi =_{df} \forall v^4(t\in v^4(\varphi)\Rightarrow t\in v^4(\psi))$.

Entailment relation so defined (call it *FDE-entailment*) is faithful to the consequence relation of $\mathbf{E}_{\mathsf{fde}}$ ($\mathbf{R}_{\mathsf{fde}}$), that is, $\mathbf{E}_{\mathsf{fde}}$ ($\mathbf{R}_{\mathsf{fde}}$) is sound and complete with respect to Definition 8.

Theorem 9. *For any* $\varphi,\psi\in\mathcal{L}_{\{\wedge,\vee,\sim\}}$, $\varphi\vdash\psi$ *is provable in* $\mathbf{E}_{\mathsf{fde}}$ *(*$\mathbf{R}_{\mathsf{fde}}$*) iff* $\varphi\vDash_{\mathsf{fde}}\psi$.

Proof. See, e.g., Dunn [34, Theorem 7]. $\lhd$

Now, consider the following consequences, which are not derivable in $\mathbf{E}_{\mathsf{fde}}$ (and, of course, neither in $\mathbf{R}_{\mathsf{fde}}$).

$$(veq) \qquad \varphi\vdash\psi\vee{\sim}\psi$$

$$(efq) \qquad \varphi\wedge{\sim}\varphi\vdash\psi$$

$$(saf) \qquad \varphi\wedge{\sim}\varphi\vdash\psi\vee{\sim}\psi$$

Principles *verum ex quodlibet* (veq), *ex falso quodlibet* (efq), and (saf) are the consequence (first-degree entailment) analogues of the Positive Paradox, Negative Paradox and Safety, respectively (in the sense outlined in footnote 1). Extending $\mathbf{E}_{\mathsf{fde}}$ or $\mathbf{R}_{\mathsf{fde}}$ with these consequences as axioms yields consequence systems for some well-known logics, including the first-degree entailment fragment of **RM**. The latter consequence system is obtained by adding (saf) either to $\mathbf{E}_{\mathsf{fde}}$ or $\mathbf{R}_{\mathsf{fde}}$, and it is labeled by $\mathbf{RM}_{\mathsf{fde}}$ in [34]. Thus, (saf) is the characteristic consequence for the first-degree entailment fragment of **R**-Mingle, and in this sense one can consider Safety to be a representative of Mingle *on the first-degree entailment level*.

Notably, if $(x)\in\{(veq),(efq),(saf)\}$, then adding (x) to $\mathbf{E}_{\mathsf{fde}}$ is not always deductively equivalent to $\mathbf{R}_{\mathsf{fde}}+(x)$. For example, $\mathbf{E}_{\mathsf{fde}}+(veq)$ is deductively equivalent to $\mathbf{E}_{\mathsf{fde}}+(efq)$, and yields a binary consequence system of classical entailment that includes all valid consequences between formulas of classical logic, cf. Shramko [58, pp. 255–256]. In contrast, $\mathbf{R}_{\mathsf{fde}}+(veq)$ is not deductively equivalent to $\mathbf{R}_{\mathsf{fde}}+(efq)$, and neither of these extensions alone produces classical consequence. To obtain the classical consequence based of $\mathbf{R}_{\mathsf{fde}}$, one must add *both* (veq) and (efq). On the other hand, $\mathbf{E}_{\mathsf{fde}}+(saf)$ *is* deductively equivalent to $\mathbf{R}_{\mathsf{fde}}+(saf)$. We thus have the following observation.

Observation 10. *Let the equality sign* $(=)$ *mean the deductive equivalence between consequence systems. Then,* $\mathbf{E}_{\mathsf{fde}} + (efq) \neq \mathbf{R}_{\mathsf{fde}} + (efq)$*,* $\mathbf{E}_{\mathsf{fde}} + (veq) \neq \mathbf{R}_{\mathsf{fde}} + (veq)$ *but* $\mathbf{E}_{\mathsf{fde}} + (saf) = \mathbf{R}_{\mathsf{fde}} + (saf)$*.*

This observation shows that systems $\mathbf{E}_{\mathsf{fde}}$ and $\mathbf{R}_{\mathsf{fde}}$, even being deductively equivalent, are not of equal strength in terms of their possible extensions. Whereas $\mathbf{E}_{\mathsf{fde}}$ allows *only two* nontrivial extensions, namely, the first-degree entailment fragment of **RM** = $\mathbf{E}_{\mathsf{fde}} + (saf)$, see, e.g., [28, p. 157 and note 7], and a system for classical consequence $\mathbf{E}_{\mathsf{fde}} + (veq)$ (or equivalently $\mathbf{E}_{\mathsf{fde}} + (efq)$), system $\mathbf{R}_{\mathsf{fde}}$, even being just another formalization of first-degree entailment, nevertheless, allows *two more* nontrivial extensions. These two additional systems (which are indistinguishable from each other and from classical logic within the deductive framework of $\mathbf{E}_{\mathsf{fde}}$) are the consequence system for Kleene's strong three-valued logic $\mathbf{R}_{\mathsf{fde}} + (efq)$, and Priest's Logic of Paradox $\mathbf{R}_{\mathsf{fde}} + (veq)$, see [34, Theorem 12].

We thus should differentiate between extending a consequence *system* and extending a *logic*. To extend a logic it is enough to add some consequence to the corresponding set of consequences, and to ensure that the resulting set is closed under reflexivity and transitivity, see Definition 3. In contrast, extending a consequence system assumes adding a consequence not derivable in this system to its axioms. Clearly, the resulting logic will be automatically closed under all of this system's primitive inference rules.

Taking into account that one and the same logic can be generated (determined) by different consequence systems, we can generalize Observation 10 by the following proposition.

Proposition 11. *Let* S_1 *and* S_2 *be two different consequence systems, which formalize one and the same logic (are deductively equivalent), and let* C *be a binary consequence, such that neither* $S_1 + C$*, nor* $S_2 + C$ *is trivial. Then logics generated by* $S_1 + C$ *and by* $S_2 + C$ *may, but* do not need *be the same.*

In the context of this Proposition, and given the fact that $\mathbf{E}_{\mathsf{fde}} + (saf) = \mathbf{R}_{\mathsf{fde}} + (saf) = \mathbf{RM}_{\mathsf{fde}}$, the following questions arise.

1. Is there another consequence system ($\mathbf{X}$) formalizing the logic of first-degree entailment, such that $\mathbf{X} + (saf) \neq \mathbf{RM}_{\mathsf{fde}}$?
2. If such a formalization exists, which is the consequence C, such that $\mathbf{X} + C = \mathbf{RM}_{\mathsf{fde}}$?
3. Provided there are such $\mathbf{X}$ and C, are there other systems between $\mathbf{X} + (saf)$ and $\mathbf{X} + C$, which embody the idea of safety from the paradoxes of relevance, and if yes, how many are they?

In the following sections, I will address these questions, by introducing the notion of a purely Tarskian consequence system and applying it to a particular construction of first-degree entailment logic.

4. A Purely Tarskian Deductive Formalization of First-degree Entailment

Consider the following precise definition of what is a binary consequence system, obtained through the notion of a binary consequence rule.

Definition 12. A *binary consequence rule* is a construction of the form

$$\frac{C_1,\ldots,C_n}{C},$$

where $C_1,\ldots,C_n,C$ are binary consequence expressions. If $n=0$, the rule is an *axiom scheme*. A binary consequence rule is *proper* iff $n \geq 1$. A *binary consequence system* is a nonempty set of binary consequence rules, of which at least one is an axiom. A binary consequence system is *proper* iff it has at least one proper binary consequence rule.

In what follows I will keep saying simply "rule" (or "inference rule") instead of "binary consequence rule," and also write rules in the form $C_1,\ldots,C_n/C$. Having a consequence system, it is important to differentiate between derivable and admissible rules of this system.

Definition 13. Rule $C_1,\ldots,C_n/C$ is *derivable* in the binary consequence system S iff there is a derivation of C in S with $C_1,\ldots,C_n$ as the premisses of this derivation.

Definition 14. Rule $C_1,\ldots,C_n/C$ is *admissible* in the binary consequence system S iff whenever all of $C_1,\ldots,C_n$ are derivable in S, then so is C.

All the primitive rules of a binary consequence system are derivable by definition. Clearly, every derivable rule is admissible, but not *vice versa*. Because adding an admissible rule to a consequence system does not change the set of derivable consequences, a binary consequence system is closed under all of its admissible rules. However, while any extension of a consequence system is closed under all derivable rules of the initial system, this is not true for admissible rules. It is possible that a rule that is admissible in a consequence system will no longer be admissible in some of its extensions. Thus, the fewer derivable rules a consequence system has, the more extensions it may allow.

To ensure that a consequence system generates a logic, one should show, in particular, that (tr) is admissible in this system. The easiest way to guarantee this is to take (tr) as a primitive rule. In fact, having transitivity as a derivable rule and reflexivity as an axiom is all that is needed for a consequence system to determine a logic.

Let me call a consequence system *purely Tarskian* iff $\varphi \vdash \varphi$ is derivable for any φ, and (tr) is *the only* primitive inference rule of this system. It may well be that a certain logic cannot be generated by a purely Tarskian consequence system. Remarkably, the logic of first-degree entailment *can* be formalized by a system of this kind. Such a system has been introduced in Shramko [61] as a *genuinely structural* binary consequence system (see also Shramko [60, pp. 1234–1235]).

System $\mathbf{FDE}_\mathrm{S}$

(di_1)	$\varphi \vdash \varphi \vee \psi$	(dco)	$\varphi \vee \psi \vdash \psi \vee \varphi$	(did)	$\varphi \vee \varphi \vdash \varphi$
(ce_1)	$\varphi \wedge \psi \vdash \varphi$	(cco)	$\varphi \wedge \psi \vdash \psi \wedge \varphi$	(cid)	$\varphi \vdash \varphi \wedge \varphi$

$(das^{\vee})$ $\quad (\varphi \vee (\psi \vee \chi)) \vee \xi \vdash ((\varphi \vee \psi) \vee \chi) \vee \xi$

$(cas^{\wedge})$ $\quad ((\varphi \wedge \psi) \wedge \chi) \wedge \xi \vdash (\varphi \wedge (\psi \wedge \chi)) \wedge \xi$

$(dis_2^{\vee\wedge})$ $\quad ((\varphi \vee (\psi \wedge \chi)) \vee \xi) \wedge \tau \vdash (((\varphi \vee \psi) \wedge (\varphi \vee \chi)) \vee \xi) \wedge \tau$

$(dis_3^{\vee\wedge})$ $\quad (((\varphi \vee \psi) \wedge (\varphi \vee \chi)) \vee \xi) \wedge \tau \vdash ((\varphi \vee (\psi \wedge \chi)) \vee \xi) \wedge \tau$

$(dis_4^{\vee\wedge})$ $(((\varphi \wedge \psi) \vee (\varphi \wedge \chi)) \vee \xi) \wedge \tau \vdash ((\varphi \wedge (\psi \vee \chi)) \vee \xi) \wedge \tau$
$(dis_5^{\vee\wedge})$ $((\varphi \wedge (\psi \vee \chi)) \vee \xi) \wedge \tau \vdash (((\varphi \wedge \psi) \vee (\varphi \wedge \chi)) \vee \xi) \wedge \tau$
$(ni^{\vee\wedge})$ $(\varphi \vee \psi) \wedge \chi \vdash ({\sim}{\sim}\varphi \vee \psi) \wedge \chi$
$(ne^{\vee\wedge})$ $({\sim}{\sim}\varphi \vee \psi) \wedge \chi \vdash (\varphi \vee \psi) \wedge \chi$
$(dm_1^{\vee\wedge})$ $({\sim}(\varphi \vee \psi) \vee \chi) \wedge \xi \vdash (({\sim}\varphi \wedge {\sim}\psi) \vee \chi) \wedge \xi$
$(dm_2^{\vee\wedge})$ $(({\sim}\varphi \wedge {\sim}\psi) \vee \chi) \wedge \xi \vdash ({\sim}(\varphi \vee \psi) \vee \chi) \wedge \xi$
$(dm_3^{\vee\wedge})$ $({\sim}(\varphi \wedge \psi) \vee \chi) \wedge \xi \vdash (({\sim}\varphi \vee {\sim}\psi) \vee \chi) \wedge \xi$
$(dm_4^{\vee\wedge})$ $(({\sim}\varphi \vee {\sim}\psi) \vee \chi) \wedge \xi \vdash ({\sim}(\varphi \wedge \psi) \vee \chi) \wedge \xi$
(tr) $\varphi \vdash \psi, \psi \vdash \chi \,/\, \varphi \vdash \chi$

In comparison to $\mathbf{E}_{\mathsf{fde}}$ and $\mathbf{R}_{\mathsf{fde}}$, this system has only one primitive inference rule (tr). To compensate for the removal of the rules (ci) and (de), and to ensure their admissibility, axioms for the mutual distributivity of disjunction and conjunction, double negation introduction and elimination, and De Morgan laws are provided with a combined disjunctive-conjunctive context of the form $(\cdots \vee \chi) \wedge \xi$. Moreover, the associativity axioms for conjunction and disjunction are four-termed (and not three-termed, as usual).

A notable feature of a purely Tarskian consequence system, and particularly system $\mathbf{FDE}_{\mathrm{S}}$, is the ability to be transformed directly into a (purely inferential) Hilbert-style system. Indeed, a binary consequence expression is nothing more than a (one-premise) Hilbertian inference rule, and (tr) can be viewed as a tool for connecting such rules in a logical derivation process. Thus, a binary consequence system with (tr) as the only binary consequence rule can be easily reshaped in a form of a Hilbert system. Such Hilbert-style systems for the first-degree entailment $\mathbf{FDE}_{\mathrm{H}}$ and a family of its extensions have been elaborated in detail in Shramko [62].

I will now reproduce a number of lemmas and theorems, proofs of which can be found in [61]. In particular, the following lemma makes derivations in $\mathbf{FDE}_{\mathrm{S}}$ more manageable, by ridding them of the disjunctive/conjunctive context if redundant, and thus, securing an unrestricted implementation of all the usual properties of first-degree entailment.

Lemma 15. *For axioms $(dis_2^{\vee\wedge})$, $(dis_3^{\vee\wedge})$, $(dis_4^{\vee\wedge})$, $(dis_5^{\vee\wedge})$, $(ni^{\vee\wedge})$, $(ne^{\vee\wedge})$, $(dm_1^{\vee\wedge})$, $(dm_2^{\vee\wedge})$, $(dm_3^{\vee\wedge})$, $(dm_4^{\vee\wedge})$ of the form $(\alpha \vee \chi) \wedge \xi \vdash (\beta \vee \chi) \wedge \xi$,*

(1) *the respective consequences (dis_2)–(dm_4) of the form $\alpha \vdash \beta$ are derivable;*
(2) *the respective dual consequences $(dis_2^{\wedge\vee})$–$(dm_4^{\wedge\vee})$ of the form $(\alpha \wedge \chi) \vee \xi \vdash (\beta \wedge \chi) \vee \xi$ are derivable;*
(3) *the respective consequences $(dis_2^{\vee})$–$(dm_4^{\vee})$ of the form $\alpha \vee \chi \vdash \beta \vee \chi$, and $(dis_2^{\wedge})$–$(dm_4^{\wedge})$ of the form $\alpha \wedge \chi \vdash \beta \wedge \chi$ are derivable.*

Moreover, standard formulations of associativity for disjunction (das) $\varphi \vee (\psi \vee \chi) \vdash (\varphi \vee \psi) \vee \chi$, and conjunction (cas) $(\varphi \wedge \psi) \wedge \chi \vdash \varphi \wedge (\psi \wedge \chi)$ are derivable as well.

Lemma 16. *All the inference rules of $\mathbf{E}_{\mathsf{fde}}$ are admissible in $\mathbf{FDE}_{\mathrm{S}}$.*

Lemma 17. *System $\mathbf{FDE}_{\mathrm{S}}$ is deductively equivalent to systems $\mathbf{E}_{\mathsf{fde}}$ and $\mathbf{R}_{\mathsf{fde}}$ in the sense that they determine the same set of provable consequences.*

And system $\mathbf{FDE}_{\mathrm{S}}$ is sound and complete with respect to Definition 8.

Theorem 18. *For any φ, ψ, $\varphi \vdash_{\mathsf{fde}} \psi \Leftrightarrow \varphi \vDash_{\mathsf{fde}} \psi$.*

It is observed in [60, p. 1237] that system $\mathbf{FDE}_{\mathrm{S}}$ provides the most suitable basis for the family of all its possible extensions. Among these extensions, there is a noteworthy subfamily that represents logics that implement a "minglish idea" to varying degrees on the first-degree entailment level. This subfamily will be discussed further in the following section.

5. Variations on Safety and the Corresponding Consequence Systems

System $\mathbf{RM}_{\mathsf{fde}}$, obtained from $\mathbf{E}_{\mathsf{fde}}$ or $\mathbf{R}_{\mathsf{fde}}$ by adding (saf) as an axiom, is indeed the first-degree entailment fragment of **RM**.

Lemma 19. *For any $\varphi, \psi \in \mathcal{L}_{\{\wedge,\vee,\sim\}}$, $\varphi \to \psi$ is provable in* **RM** *iff $\varphi \vdash \psi$ is provable in $\mathbf{E}_{\mathsf{fde}} + (saf)$, or equivalently in $\mathbf{R}_{\mathsf{fde}} + (saf)$.*

The proof of this lemma will be given in Section 7. It might appear that adding (saf) to *any* (deductively equivalent) formalization of the first-degree entailment will produce the same result. However, this would be a mistake, as Rivieccio has shown in [54, p. 328] by the case study of certain extensions of the Set-Fmla system $\vdash_H$ introduced in [38]. Namely, consequence $(\varphi \wedge \sim\varphi) \vee \chi \vdash (\psi \vee \sim\psi) \vee \chi$ turns out not to be derivable in $\vdash_H +(saf)$, although it is valid in $\mathbf{RM}_{\mathsf{fde}}$. This is because the disjunction elimination rule (i.e., (de) of $\mathbf{E}_{\mathsf{fde}}$), which is admissible in $\vdash_H$, is not admissible in $\vdash_H +(saf)$. In fact, it is proved in [1, p. 1066] that adding $(\varphi \wedge \sim\varphi) \vee \chi \vdash (\psi \vee \sim\psi) \vee \chi$ to $\vdash_H$ provides a correct deductive formalization of the Set-Fmla analogue of $\mathbf{RM}_{\mathsf{fde}}$.[7]

In the case of the first-degree entailment constructed in the Fmla-Fmla framework, the situation can be even more intricate. $\mathbf{FDE}_{\mathrm{S}}$ is closed under both *disjunction elimination* (de) and *conjunction introduction* (ci) (see Lemma 16), but its various extensions may no longer be so. This allows us to distinguish *four* different versions of Safety, which correspond to the situations, where (1) neither of the two closures holds, (2)–(3) one of them holds, but the other does not, and (4) both of them hold. To express these situations by means of different deductive systems we will need the following four consequences, neither of which is derivable in $\mathbf{FDE}_{\mathrm{S}}$.

(saf) $\quad \varphi \wedge \sim\varphi \vdash \psi \vee \sim\psi$
$(saf^{\wedge})$ $\quad (\varphi \wedge \sim\varphi) \wedge \chi \vdash (\psi \vee \sim\psi) \wedge \chi$
$(saf^{\vee})$ $\quad (\varphi \wedge \sim\varphi) \vee \chi \vdash (\psi \vee \sim\psi) \vee \chi$
$(saf^{\vee\wedge})$ $\quad ((\varphi \wedge \sim\varphi) \vee \chi) \wedge \xi \vdash ((\psi \vee \sim\psi) \vee \chi) \wedge \xi$

That is, side by side with the standard principle of Safety (saf), we can consider other three versions of it, augmented with a conjunctive context, a disjunctive context, and a combined disjunctive-conjunctive context.[8] This opens the way for the corresponding binary consequence systems, which formalize different variations of Safety on the basis of $\mathbf{FDE}_{\mathrm{S}}$. Let me call these systems "FDE-based mingle logics."[9] We thus have:

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[7] Cf. also Přenosil [52, p. 12], where it is observed that instead of $(\varphi \wedge \sim\varphi) \vee \chi \vdash (\psi \vee \sim\psi) \vee \chi$ one can likewise take a consequence in two variables, namely, $(\varphi \wedge \sim\varphi) \vee \psi \vdash \psi \vee \sim\psi$.

[8] I take this opportunity to correct an erroneous formulation of $(saf^{\vee\wedge})$ in [60, p. 1238] and [62].

[9] If the term "mingle logic" is taken literally, it means that the corresponding system includes Mingle as one of its axioms. In view of this, it might be more appropriate to call these systems "safety logics." It should be noted, however, that we are dealing with the first-degree entailment framework, and Safety is

$$\mathbf{SM}_{\mathrm{S}} = \mathbf{FDE}_{\mathrm{S}} + (\mathit{saf}) \qquad \mathbf{RM}^{\wedge}_{\mathrm{S}} = \mathbf{FDE}_{\mathrm{S}} + (\mathit{saf}^{\wedge})$$
$$\mathbf{RM}^{\vee}_{\mathrm{S}} = \mathbf{FDE}_{\mathrm{S}} + (\mathit{saf}^{\vee}) \qquad \mathbf{RM}_{\mathrm{S}} = \mathbf{FDE}_{\mathrm{S}} + (\mathit{saf}^{\vee\wedge})$$

$\mathbf{SM}_{\mathrm{S}}$ stands for "sub-mingle," and it is the weakest FDE-based logic that validates Safety, the characteristic principle of a mingle logic on the first-degree entailment level. It is not, however, the first-degree entailment fragment of **RM**, because some consequences, whose implicational counterparts are derivable in **RM**, are not derivable in $\mathbf{SM}_{\mathrm{S}}$. In particular, $(\mathit{saf}^{\wedge})$, $(\mathit{saf}^{\vee})$ and $(\mathit{saf}^{\vee\wedge})$ do not hold in $\mathbf{SM}_{\mathrm{S}}$, although $(\varphi \wedge\sim\varphi)\wedge\chi \to (\psi\vee\sim\psi)\wedge\chi$, $(\varphi\wedge\sim\varphi)\vee\chi \to (\psi\vee\sim\psi)\vee\chi$ and $((\varphi\wedge\sim\varphi)\vee\chi)\wedge\xi \to ((\psi\vee\sim\psi)\vee\chi)\wedge\xi$ are theorems of **RM**. This enables further unrolling of Safety in two different directions by means of $\mathbf{RM}^{\wedge}_{\mathrm{S}}$ and $\mathbf{RM}^{\vee}_{\mathrm{S}}$. The first of these two systems turns out to be an intermediate stage on the road to Pietz/Kapsner and Rivieccio's "Exactly True Logic" [51], whereas the second one leads to "Non-Falsity Logic" introduced in Shramko et al. [63], see in more detail in [62]. The union of $\mathbf{RM}^{\wedge}_{\mathrm{S}}$ and $\mathbf{RM}^{\vee}_{\mathrm{S}}$ results in $\mathbf{RM}_{\mathrm{S}}$, which is exactly the first-degree entailment fragment of **RM** (the proof of this fact will be given in Section 7 along with the proof of Lemma 19).

It is not difficult to extend (1)–(3) from Lemma 15 to the case with $(\mathit{saf}^{\vee\wedge})$, and thus to show that (saf) is derivable in all FDE-based mingle logics, whereas both $(\mathit{saf}^{\vee})$ and $(\mathit{saf}^{\wedge})$ are derivable in $\mathbf{RM}_{\mathrm{S}}$. As an example, consider the derivation of (saf) in $\mathbf{RM}^{\vee}_{\mathrm{S}}$.

1. $\varphi\wedge\sim\varphi \vdash (\varphi\wedge\sim\varphi)\vee(\psi\vee\sim\psi)$ $\quad (di_1)$
2. $(\varphi\wedge\sim\varphi)\vee(\psi\vee\sim\psi) \vdash (\psi\vee\sim\psi)\vee(\psi\vee\sim\psi)$ $\quad (\mathit{saf}^{\vee})$
3. $(\psi\vee\sim\psi)\vee(\psi\vee\sim\psi) \vdash \psi\vee\sim\psi$ $\quad (did)$
4. $\varphi\wedge\sim\varphi \vdash \psi\vee\sim\psi$ $\quad$ (1–3; (tr), twice)

Relations between the FDE-based mingle systems are such that they constitute a four-element lattice presented in Figure 1 (together with the base system $\mathbf{FDE}_{\mathrm{S}}$). In this Hasse diagram, the order is the subset relation between the sets of provable consequences of the systems. It is also a sublattice of the lattice of FDE family constructed in [60, Figure 1] (cf. Figure 4 in [62]).

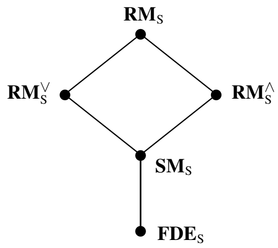

FIGURE 1. Diamond of FDE-based mingle logics

Remarkably, the four system defined above are not the only FDE-based mingle logics. In fact, one can show that there are *infinitely many* of such logics between $\mathbf{SM}_{\mathrm{S}}$ and $\mathbf{RM}^{\wedge}_{\mathrm{S}}$, as well as between $\mathbf{RM}^{\wedge}_{\mathrm{S}}$ and $\mathbf{RM}_{\mathrm{S}}$ on the one side, and also between

the characteristic principle of the first-degree entailment fragment of the basic mingle logic **RM**. Therefore, I employ the label which reflects the interconnection between Mingle and Safety on the first-degree entailment level.

$\mathbf{SM}_\mathrm{S}$ and $\mathbf{RM}^{\vee}_\mathrm{S}$, as well as between $\mathbf{RM}^{\vee}_\mathrm{S}$ and $\mathbf{RM}_\mathrm{S}$ on the other side. To show this, consider the following lemma.

Lemma 20. *Rules* (ci) *and* (de) *of the system* $\mathbf{E}_{\mathsf{fde}}$ *are not admissible in* $\mathbf{SM}_\mathrm{S}$*;* (ci) *is admissible in* $\mathbf{RM}^{\wedge}_\mathrm{S}$*, but not in* $\mathbf{RM}^{\vee}_\mathrm{S}$*;* (de) *is admissible in* $\mathbf{RM}^{\vee}_\mathrm{S}$*, but not in* $\mathbf{RM}^{\wedge}_\mathrm{S}$*; and both* (ci) *and* (de) *are admissible in* $\mathbf{RM}_\mathrm{S}$*.*

Proof. To see that neither (ci), nor (de) is admissible in $\mathbf{SM}_\mathrm{S}$, note that the following consequences are not derivable in this system

$$(saf_{\wedge 1}) \qquad (\varphi\wedge\sim\varphi)\vdash(\psi\vee\sim\psi)\wedge(\chi\vee\sim\chi),$$

$$(saf_{\vee 1}) \qquad (\varphi\wedge\sim\varphi)\vee(\psi\wedge\sim\psi)\vdash(\chi\vee\sim\chi),$$

although $\varphi\wedge\sim\varphi\vdash\chi\vee\sim\chi$, $\psi\wedge\sim\psi\vdash\chi\vee\sim\chi$, $\varphi\wedge\sim\varphi\vdash\psi\vee\sim\psi$ are derivable in it. The mentioned non-derivability can be established semantically by using the completeness result from Theorem 23 in the next section. I postpone this task till then.

Moreover, (ci) is not admissible in $\mathbf{RM}^{\vee}_\mathrm{S}$, because $(saf_{\wedge 1})$ is not $\mathbf{RM}^{\vee}_\mathrm{S}$-derivable, and (de) is not admissible in $\mathbf{RM}^{\wedge}_\mathrm{S}$, because $(saf_{\vee 1})$ is not $\mathbf{RM}^{\wedge}_\mathrm{S}$-derivable.

To see that both (ci) and (de) are admissible in $\mathbf{RM}_\mathrm{S}$, it is enough to show that the following consequences are derivable in it (cf. [62, Lemma 4.2]).

$$(((\varphi\wedge\sim\varphi)\vee\chi)\wedge\xi)\wedge\upsilon\vdash(((\psi\vee\sim\psi)\vee\chi)\wedge\xi)\wedge\upsilon,$$
$$(((\varphi\wedge\sim\varphi)\vee\chi)\wedge\xi)\vee\upsilon\vdash(((\psi\vee\sim\psi)\vee\chi)\wedge\xi)\vee\upsilon,$$

which is not a difficult exercise.

Analogously, to see that (ci) is admissible in $\mathbf{RM}^{\wedge}_\mathrm{S}$ it is enough to observe that $((\varphi\wedge\sim\varphi)\wedge\chi)\wedge\xi\vdash((\psi\vee\sim\psi)\wedge\chi)\wedge\xi$ is derivable in it, whereas for the admissibility of (de) in $\mathbf{RM}^{\vee}_\mathrm{S}$, one has to show the derivability of $((\varphi\wedge\sim\varphi)\vee\chi)\vee\xi\vdash((\psi\vee\sim\psi)\vee\chi)\vee\xi$. $\lhd$

In view of Lemma 20, we can apply the methodology from [54], and first observe, that $\mathbf{SM}_\mathrm{S}$ turns out to be stronger than the system $\mathbf{SM}_{\mathrm{S}\wedge 1}=\mathbf{SM}_\mathrm{S}+(saf_{\wedge 1})$, in the sense that the set of consequences derivable in the first system is a proper subset of the set of consequences derivable in the second system, i.e., $\mathbf{SM}_\mathrm{S}\subset\mathbf{SM}_{\wedge 1\mathrm{S}}$. It is possible to generalize this observation by considering the consequences

$$(saf_{\wedge n}) \qquad \varphi\wedge\sim\varphi\vdash(\psi_0\vee\sim\psi_0)\wedge\cdots\wedge(\psi_n\vee\sim\psi_n)$$

for each $n\geq 1$. The corresponding systems are then defined as $\mathbf{SM}_{\mathrm{S}\wedge n}=\mathbf{SM}_\mathrm{S}+(saf_{\wedge n})$. We obtain then a denumerable chain of extensions of $\mathbf{SM}_\mathrm{S}$:

$$\mathbf{SM}_\mathrm{S}\subset\mathbf{SM}_{\mathrm{S}\wedge 1}\subset\cdots\subset\mathbf{SM}_{\mathrm{S}\wedge n}\subset\cdots\subset\mathbf{SM}_{\mathrm{S}\wedge\infty}\subset\mathbf{RM}^{\wedge}_\mathrm{S},$$

such that $\mathbf{SM}_{\mathrm{S}\wedge n}\subset\mathbf{SM}_{\mathrm{S}\wedge n+1}$ for any $n\geq 1$, and $\mathbf{SM}_{\mathrm{S}\wedge\infty}$ is the union of all the elements of the chain, except of $\mathbf{RM}^{\wedge}_\mathrm{S}$ (cf. [54, p. 330]).

Furthermore, since $\mathbf{RM}^{\wedge}_\mathrm{S}$ is not closed under (de), it turns out that, e.g.,

$$(saf^{\wedge}_{\vee 1}) \qquad ((\varphi_0\wedge\sim\varphi_0)\vee(\varphi_1\wedge\sim\varphi_1))\wedge\chi\vdash(\psi\vee\sim\psi)\wedge\chi$$

is not derivable in $\mathbf{RM}^{\wedge}_\mathrm{S}$, although both $(\varphi_0\wedge\sim\varphi_0)\wedge\chi\vdash(\psi\vee\sim\psi)\wedge\chi$ and $(\varphi_1\wedge\sim\varphi_1)\wedge\chi\vdash(\psi\vee\sim\psi)\wedge\chi$ are derivable in it. Generalizing this observation, consider the consequences

$$(saf^{\wedge}_{\vee n}) \qquad ((\varphi_0\wedge\sim\varphi_0)\vee\cdots\vee(\varphi_n\wedge\sim\varphi_n))\wedge\chi\vdash(\psi\vee\sim\psi)\wedge\chi$$

for each $n \geq 1$, and define the corresponding systems $\mathbf{RM}^{\wedge}_{\mathrm{S}\vee n} = \mathbf{RM}^{\wedge}_{\mathrm{S}} + (saf^{\wedge}_{\vee n})$. We then again obtain a denumerable chain of extensions of $\mathbf{RM}^{\wedge}_{\mathrm{S}}$:

$$\mathbf{RM}^{\wedge}_{\mathrm{S}} \subset \mathbf{RM}^{\wedge}_{\mathrm{S}\vee 1} \subset \cdots \subset \mathbf{RM}^{\wedge}_{\mathrm{S}\vee n} \subset \cdots \subset \mathbf{RM}^{\wedge}_{\mathrm{S}\vee\infty} \subset \mathbf{RM}_{\mathrm{S}},$$

such that $\mathbf{RM}^{\wedge}_{\mathrm{S}\vee n} \subset \mathbf{RM}^{\wedge}_{\mathrm{S}\vee n+1}$ for any $n \geq 1$.

Dually, for each $n \geq 1$, we can first consider the consequence

$$(saf_{\vee n}) \qquad (\varphi_0 \wedge \sim \varphi_0) \vee \cdots \vee (\varphi_n \wedge \sim \varphi_n) \vdash \psi \vee \sim \psi,$$

and the corresponding system $\mathbf{SM}_{\mathrm{S}\vee n} = \mathbf{SM}_{\mathrm{S}} + (saf_{\vee n})$. A denumerable chain of extensions of $\mathbf{SM}_{\mathrm{S}}$ in other direction looks then as follows:

$$\mathbf{SM}_{\mathrm{S}} \subset \mathbf{SM}_{\mathrm{S}\vee 1} \subset \cdots \subset \mathbf{SM}_{\mathrm{S}\vee n} \subset \cdots \subset \mathbf{SM}_{\mathrm{S}\vee\infty} \subset \mathbf{RM}^{\vee}_{\mathrm{S}},$$

such that $\mathbf{SM}_{\mathrm{S}\vee n} \subset \mathbf{SM}_{\mathrm{S}\vee n+1}$ for any $n \geq 1$. Moving on this path further towards $\mathbf{RM}_{\mathrm{S}}$, we can consider the consequences:

$$(saf^{\vee}_{\wedge n}) \qquad (\varphi \wedge \sim \varphi) \vee \chi \vdash ((\psi_0 \vee \sim \psi_0) \wedge \cdots \wedge (\psi_n \vee \sim \psi_n)) \vee \chi$$

for each $n \geq 1$, and define the corresponding systems $\mathbf{RM}^{\vee}_{\mathrm{S}\wedge n} = \mathbf{RM}^{\vee}_{\mathrm{S}} + (saf^{\vee}_{\wedge n})$. We then obtain a denumerable chain of extensions of $\mathbf{RM}^{\vee}_{\mathrm{S}}$:

$$\mathbf{RM}^{\vee}_{\mathrm{S}} \subset \mathbf{RM}^{\vee}_{\mathrm{S}\wedge 1} \subset \cdots \subset \mathbf{RM}^{\vee}_{\mathrm{S}\wedge n} \subset \cdots \subset \mathbf{RM}^{\vee}_{\mathrm{S}\wedge\infty} \subset \mathbf{RM}_{\mathrm{S}},$$

such that $\mathbf{RM}^{\vee}_{\mathrm{S}\wedge n} \subset \mathbf{RM}^{\vee}_{\mathrm{S}\wedge n+1}$ for any $n \geq 1$.

Thus, each side of the diamond in Figure 1 contains infinitely many FDE-based mingle systems. Algebraic properties of these chains of systems deserve special consideration.

6. Semantics for Safety

Belnap, when explaining his useful four-valued logic of how a computer should think, argues, in particular, that a computer should "say that the inference from A to B is valid, or that A entails B, if the inference never leads us from the True to the absence of the True (preserves Truth), *and also* never leads us from the absence of the False to the False (preserves non-Falsity)", see in [49, p. 44, emphasis in original]. He immediately adds: "Dunn, 1976, has shown that it suffices to mention truth-preservation, since if some inference form fails to always preserve non-Falsity, then it can be shown by a technical argument that it also fails to preserve Truth" [ibid.]. The following lemma confirms the latter remark with respect to the Dunn–Belnap intuitive semantics for first-degree entailment.

Lemma 21. *For any* $\varphi, \psi \in \mathcal{L}_{\{\wedge,\vee,\sim\}}$, $\varphi \vDash_{\mathsf{fde}} \psi \Leftrightarrow \forall v (f \in v^4(\psi) \Rightarrow f \in v^4(\varphi))$.

Proof. See, e.g., Dunn [34, Proposition 4]. ◁

Belnap interprets this result of Dunn in the sense that "the False really is on all fours with the True, so that it is profoundly natural to state our account of 'valid' or 'acceptable' inference in a way which is neutral with respect to the two" [ibid.]. Now, it can also make sense to focus on the emphasis in the initial quotation. Why not interpret it as meaning that the conditions of truth preservation *and* non-falsity preservation are not interchangeable, but rather mutually complementary (and thus jointly necessary)? To be specific, one can assume, that, at least *in some cases*, it is not sufficient for a valid inference to preserve truth *or* (equivalently) non-falsity, as is the case with FDE-entailment, but that *both* preservations must be used as essential components of an entailment framework.

Quite notably, if we will get literal with Belnap's initial suggestion that a valid inference should *explicitly* preserve *both* truth and non-falsity, we can obtain semantics for various FDE-based mingle systems. Dunn in [28, Note 7] explains that we "can capture semantically the first-degree implications of the system **RM**" (for which, he says, (*saf*) is "kind of characteristic") when, together with the usual requirement of truth preservation, we "bring into the definition of 'validity' the *additional* requirement that whenever the conclusion is false so is the premiss" (italics mine). Moreover, it turns out that if by these requirements we will appropriately incorporate two types of truth and falsity — to be *only* true (false), and to be *at least* true (false), then we can develop a general semantic framework which covers all the four "corner systems" of our Diamond. Whereas truth and falsity conditions for $\mathbf{RM}_\mathrm{S}$ can be defined in terms of "at least values" (see [34, p. 15]), and truth and falsity conditions for $\mathbf{SM}_\mathrm{S}$ employ "only values" (see [62, p. 19]), the semantics for $\mathbf{RM}_\mathrm{S}^\wedge$ and $\mathbf{RM}_\mathrm{S}^\vee$ can be constructed as certain combinations thereof.

Recall generalized truth-value function v^4, which has been defined in Section 3 as a map from the set of propositional variables ($\mathcal{V}ar$) into Dunn–Belnap's set of (four) generalized truth values $\{T,F,B,N\}$. Define a three-valued truth-value function on a subset of the set of four truth values, namely, $v^3 \colon \mathcal{V}ar \mapsto \{T,F,N\}$, and extend it to compound formulas as in Definition 7, *mutatis mutandis*. We have then the following definition of entailment relations for the four main FDE-based mingle logics.

Definition 22.
1. $\varphi \vDash_{\mathsf{sm}} \psi =_{df} \forall v^4(v^4(\varphi) = T \Rightarrow v^4(\psi) = T)\ \&\ \forall v^4(v^4(\psi) = F \Rightarrow v^4(\varphi) = F)$;
2. $\varphi \vDash_{\mathsf{rm}^\wedge} \psi =_{df} \forall v^4(v^4(\varphi) = T \Rightarrow v^4(\psi) = T)\ \&\ \forall v^3(f \in v^3(\psi) \Rightarrow f \in v^3(\varphi))$;
3. $\varphi \vDash_{\mathsf{rm}^\vee} \psi =_{df} \forall v^3(t \in v^3(\varphi) \Rightarrow t \in v^3(\psi))\ \&\ \forall v^4(v^4(\psi) = F \Rightarrow v^4(\varphi) = F)$;
4. $\varphi \vDash_{\mathsf{rm}} \psi =_{df} \forall v^3(t \in v^3(\varphi) \Rightarrow t \in v^3(\psi))\ \&\ \forall v^3(f \in v^3(\psi) \Rightarrow f \in v^3(\varphi))$.

The entailment relation of $\mathbf{SM}_\mathrm{S}$ ensures the preservation of value T from a premise to a conclusion, *and also* the preservation of value F in the backward direction (from conclusion to premise) in the Dunn–Belnap four-valued framework. Thus, T plays here the role of the *designated* truth value, whereas F can be considered the *antidesignated* one, cf. [64, p. 492]. The definition of entailment relation in $\mathbf{RM}_\mathrm{S}$ retains the same designated and antidesignated truth values, but now in a three-valued setting $\{T,F,N\}$.[10] It is also noteworthy that if in the definition of $\vDash_{\mathsf{sm}}$ we keep just the first part, which deals with the value T, we obtain semantics for the Exactly True Logic from [51], and if we focus only on the second part with the value F, the result will be semantics for the Non-Falsity Logic from [63]. Quite remarkably, semantics for Kleene's strong three-valued logic and Priest's Logic of Paradox are obtained in the same way from the definition of $\vDash_{\mathsf{rm}}$, see [62, Definition 5.1]. Therefore, relations of $\mathbf{SM}_\mathrm{S}$ to Exactly True Logic and Non-Falsity Logic are exactly the same as the relations of $\mathbf{RM}_\mathrm{S}$ to Kleene's logic and Priest's logic.

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[10] Observe, that it could equivalently be defined by means of another truth-value function $v^{3'} \colon \mathcal{V}ar \mapsto \{T,F,B\}$, cf. [34, Theorem 12 (*iii*)].

Now, since $\mathbf{RM}^{\wedge}_{\mathrm{S}}$ is the intersection of $\mathbf{RM}_{\mathrm{S}}$ and Exactly True Logic, the definition of $\vDash_{\mathsf{rm}^{\wedge}}$ is in fact the combination of definitions of their entailment relations. Note, that in the definition of $\vDash_{\mathsf{rm}^{\wedge}}$ we use simultaneously two truth-value functions, v^4 and v^3. The definition of $\vDash_{\mathsf{rm}^{\vee}}$ is obtained dually.

In what follows, I will denote by $\varphi \vdash_{\mathsf{sm}} \psi$, $\varphi \vdash_{\mathsf{rm}^{\wedge}} \psi$, $\varphi \vdash_{\mathsf{rm}^{\vee}} \psi$ and $\varphi \vdash_{\mathsf{rm}} \psi$ the facts that the consequence $\varphi \vdash \psi$ is provable in the system $\mathbf{SM}_{\mathrm{S}}$, $\mathbf{RM}^{\wedge}_{\mathrm{S}}$, $\mathbf{RM}^{\vee}_{\mathrm{S}}$ and $\mathbf{RM}_{\mathrm{S}}$ respectively. We have then the following soundness theorem.

Theorem 23. *Let* s *be* sm, $\mathsf{rm}^{\wedge}$, $\mathsf{rm}^{\vee}$ *or* rm. *Then* $\varphi \vdash_{\mathsf{s}} \psi \Rightarrow \varphi \vDash_{\mathsf{s}} \psi$.

Proof. Let us check only the characteristic consequences of each system.

1. Consider (*saf*) and $\vDash_{\mathsf{sm}}$. Assume, that $\varphi \wedge \sim\varphi \nvDash_{\mathsf{sm}} \psi \vee \sim\psi$. Then, (a) there is v^4, such that $[v^4(\varphi \wedge \sim\varphi) = T$, and $v^4(\psi \vee \sim\psi) \neq T]$; or (b) there is v^4, such that $[v^4(\psi \vee \sim\psi) = F$, and $v^4(\varphi \wedge \sim\varphi) \neq F]$. In the case of (a) we have, in particular, $v^4(\varphi) = T$, and $v^4(\varphi) = F$, which is impossible. In the case of (b), we have that $v^4(\psi) = F$ and $v^4(\psi) = T$, which is again impossible.

2. Consider ($saf^{\wedge}$) and $\vDash_{\mathsf{rm}^{\wedge}}$. Assume $(\varphi \wedge \sim\varphi) \wedge \chi \nvDash_{\mathsf{rm}^{\wedge}} (\psi \vee \sim\psi) \wedge \chi$. Then at least one of the following two cases should be possible:

(a) $\exists v^4[v^4((\varphi \wedge \sim\varphi) \wedge \chi) = T$ and $v^4((\psi \vee \sim\psi) \wedge \chi) \neq T]$.

(b) $\exists v^3[f \in v^3((\psi \vee \sim\psi) \wedge \chi)$ and $f \notin v^3((\varphi \wedge \sim\varphi) \wedge \chi)]$.

Take (a). We immediately get $v^4(\varphi) = T$, and $v^4(\varphi) = F$, which is impossible.

Take (b). Then $[(f \in v^3(\psi)$ and $t \in v^3(\psi))$, or $f \in v^3(\chi)]$, and $f \notin v^3(\varphi)$, $t \notin v^3(\varphi)$, $f \notin v^3(\chi)$. In the first case we have $f \in v^3(\psi)$, $t \in v^3(\psi)$, which is impossible for v^3. In the second case we obtain $f \in v^3(\chi)$ and $f \notin v^3(\chi)$, a contradiction.

3. Consider ($saf^{\vee}$) and $\vDash_{\mathsf{rm}^{\vee}}$. Assume $(\varphi \wedge \sim\varphi) \vee \chi \nvDash_{\mathsf{rm}^{\vee}} (\psi \vee \sim\psi) \vee \chi$. Then at least one of the following two cases should be possible:

(a) $\exists v^3[t \in v^3((\varphi \wedge \sim\varphi) \vee \chi)$ and $t \notin v^3((\psi \vee \sim\psi) \vee \chi)]$.

(b) $\exists v^4[v^4((\psi \vee \sim\psi) \vee \chi) = F$ and $v^4((\varphi \wedge \sim\varphi) \vee \chi) \neq F]$.

Take (a). Then $[(t \in v^3(\varphi)$ and $f \in v^3(\varphi))$, or $t \in v^3(\chi)]$, and $t \notin v^3(\psi)$, $f \notin v^3(\psi)$, $t \notin v^3(\chi)$. In the first case, we get $t \in v^3(\varphi)$, and $f \in v^3(\varphi)$, which is impossible for v^3. In the second case, we obtain $t \in v^3(\chi)$, and $t \notin v^3(\chi)$, a contradiction.

Take (b). We immediately get $v^4(\psi) = F$, and $v^4(\psi) = T$, which is impossible.

4. Consider ($saf^{\vee\wedge}$) and $\vDash_{\mathsf{rm}}$. Assume $((\varphi \wedge \sim\varphi) \vee \chi) \wedge \xi \nvDash_{\mathsf{rm}} ((\psi \vee \sim\psi) \vee \chi) \wedge \xi$. Then at least one of the following two cases should be possible:

(a) $\exists v^3[t \in v^3(((\varphi \wedge \sim\varphi) \vee \chi) \wedge \xi)$, and $t \notin v^3(((\psi \vee \sim\psi) \vee \chi) \wedge \xi)]$.

(b) $\exists v^3[f \in v^3(((\psi \vee \sim\psi) \vee \chi) \wedge \xi)$, and $f \notin v^3(((\varphi \wedge \sim\varphi) \vee \chi) \wedge \xi)]$.

Take (a). Then, first, $t \in v^3((\varphi \wedge \sim\varphi) \vee \chi)$, and $t \in v^3(\xi)$, whereas $t \notin v^3((\psi \vee \sim\psi) \vee \chi)$, or $t \notin v^3(\xi)$. We rule out the case $t \in v^3(\xi)$, and $t \notin v^3(\xi)$ (a contradiction), and what remains is: $[(t \in v^3(\varphi)$, and $f \in v^3(\varphi))$, or $t \in v^3(\chi)]$, and $t \notin v^3(\psi)$, $f \notin v^3(\psi)$, $t \notin v^3(\chi)$. In the first case, we get $t \in v^3(\varphi)$, and $f \in v^3(\varphi)$, which is impossible for v^3, and in the second case, we obtain $t \in v^3(\chi)$, and $t \notin v^3(\chi)$, a contradiction.

Case (b) is covered analogously. $\lhd$

Having Theorem 23, one can show, as promised, that, e.g., ($saf_{\wedge 1}$) from the proof of Lemma 20 is not derivable in $\mathbf{SM}_{\mathrm{S}}$. Indeed, let, e.g., $v^4(\varphi) = B$, $v^4(\psi) = B$ and $v^4(\chi) = N$. In this case $v^4((\psi \vee \sim\psi) \wedge (\chi \vee \sim\chi)) = F$, but $v^4(\varphi \wedge \sim\varphi) = B$. Hence, $(\varphi \wedge \sim\varphi) \nvDash_{\mathsf{sm}} (\psi \vee \sim\psi) \wedge (\chi \vee \sim\chi)$.

For a completeness proof, we employ the canonical model constructions in terms of theories. For any system S we as usual define *S-theory* $\mathcal{T}$ as the set of formulas closed under $\vdash_{\mathsf{s}}$ and conjunction introduction, that is, $\varphi \in \mathcal{T}, \varphi \vdash_{\mathsf{s}} \psi \Rightarrow \psi \in \mathcal{T}$ and $\varphi \in \mathcal{T}, \psi \in \mathcal{T} \Rightarrow \varphi \wedge \psi \in \mathcal{T}$. A theory is *prime* iff it has the disjunction property, that is, $\varphi \vee \psi \in \mathcal{T} \Rightarrow \varphi \in \mathcal{T}$ or $\psi \in \mathcal{T}$. A theory $\mathcal{T}$ is *consistent* iff there is no φ, such that both $\varphi \in \mathcal{T}$ and $\sim\varphi \in \mathcal{T}$. $\mathcal{T}$ is *decisive* iff for each φ, $\varphi \in \mathcal{T}$ or $\sim\varphi \in \mathcal{T}$.

We first prove the completeness of $\mathbf{RM}_{\mathrm{S}}$ by considering **RM**-theories. The following variation of Lindenbaum's lemma holds:

Lemma 24. *If $\varphi \nvdash_{\mathsf{rm}} \psi$, then there is a consistent prime* **RM***-theory $\mathcal{T}$, such that $\varphi \in \mathcal{T}$ and $\psi \notin \mathcal{T}$, or there is a consistent prime* **RM***-theory $\mathcal{T}$, such that $\sim\varphi \notin \mathcal{T}$ and $\sim\psi \in \mathcal{T}$.*

Proof. As usual, one starts from the set of formulas $\mathcal{T}_0 = \{\psi' \colon \varphi \vdash_{\mathsf{rm}} \psi'\}$. It is easy to see that $\mathcal{T}_0$ is an **RM**-theory (since (tr) and (ci) are admissible in $\mathbf{RM}_{\mathrm{S}}$), and moreover, $\varphi \in \mathcal{T}_0$, and $\psi \notin \mathcal{T}_0$. Enumerate all the sentences of our language $\chi_0, \chi_1, \chi_2, \ldots$ (where χ_0 is φ), and consider a series of **RM**-theories $\mathcal{T}_0, \mathcal{T}_1, \mathcal{T}_2, \ldots$, defining for any $\mathcal{T}_n$ theory $\mathcal{T}_{n+1}$ as follows: (1) if $\psi \notin \mathcal{T}_n + \varphi_n$, then $\mathcal{T}_{n+1} = \mathcal{T}_n + \varphi_n$; (2) $\mathcal{T}_{n+1} = \mathcal{T}_n$, otherwise. Consider the union of all $\mathcal{T}_n$. It is easy to see that $\mathcal{T}$ is an **RM**-theory, such that $\varphi \in \mathcal{T}$ and $\psi \notin \mathcal{T}$. By using the closure of **RM**-theories under (de) and distributivity rules one can show that $\mathcal{T}$ is also prime, cf. e.g., Dunn [34, Lemma 8].

Now, if this theory is consistent, we are through. If it is inconsistent, then by (saf), which is derivable in $\mathbf{RM}_{\mathrm{S}}$, it is decisive. Define theory $\mathcal{T}^*$ as follows (for any χ): (1) $\chi \in \mathcal{T}^* \Leftrightarrow \sim\chi \notin \mathcal{T}$; (2) $\sim\chi \in \mathcal{T}^* \Leftrightarrow \chi \notin \mathcal{T}$. Using the closure of $\mathbf{RM}_{\mathrm{S}}$ under (con), it is not difficult to show that $\mathcal{T}^*$ is indeed a prime **RM**-theory, which is consistent. We also have $\sim\varphi \notin \mathcal{T}^*$ and $\sim\psi \in \mathcal{T}^*$. ◁

We now have the following valuation lemma.

Lemma 25. *Let $\mathcal{T}$ be a prime* **RM***-theory, and define a canonical valuation v_τ so that $t \in v_\tau(p)$ iff $p \in \mathcal{T}$, and $f \in v_\tau(p)$ iff $\sim p \in \mathcal{T}$. Then truth and falsity conditions of compound formulas (Definition 7) hold for the canonical valuation so defined.*

Proof. A simple check. ◁

Theorem 26. *For any $\varphi, \psi \in \mathcal{L}_{\{\wedge,\vee,\sim\}}$, $\varphi \vDash_{\mathsf{rm}} \psi \Rightarrow \varphi \vdash_{\mathsf{rm}} \psi$.*

Proof. Let $\varphi \nvdash_{\mathsf{rm}} \psi$. By Lemma 24 there is a consistent prime **RM**-theory $\mathcal{T}$, such that $\varphi \in \mathcal{T}$ and $\psi \notin \mathcal{T}$, or there is a consistent prime **RM**-theory $\mathcal{T}$, such that $\sim\varphi \notin \mathcal{T}$ and $\sim\psi \in \mathcal{T}$. Thus, there is a canonical valuation v_τ, such that $t \in v_\tau(\varphi)$, and $t \notin v_\tau(\psi)$, or there is a canonical valuation v_τ, such that $f \notin v_\tau(\varphi)$, and $f \in v_\tau(\psi)$. Consistency of $\mathcal{T}$ ensures that v_τ is v^3. Hence, $\varphi \nvDash_{\mathsf{rm}} \psi$. ◁

For the completeness proof of $\mathbf{SM}_{\mathrm{S}}$, $\mathbf{RM}_{\mathrm{S}}^{\wedge}$, and $\mathbf{RM}_{\mathrm{S}}^{\vee}$, we will deal with FDE-theories to obtain the corresponding variations of the Lindenbaum lemma.

Lemma 27. *Let $\varphi \nvdash_{\mathsf{sm}} \psi$. Then there is a prime FDE-theory $\mathcal{T}$, such that $\varphi \in \mathcal{T}$, $\sim\varphi \notin \mathcal{T}$, and $\psi \notin \mathcal{T}$, or there is a prime FDE-theory $\mathcal{T}$, such that $\sim\varphi \notin \mathcal{T}$, $\sim\psi \in \mathcal{T}$, and $\psi \notin \mathcal{T}$.*

Proof. First, take as the starting theory $\mathcal{T}_0 = \{\psi' : \varphi \vdash_{\mathsf{fde}} \psi'\}$. We have $\varphi \in \mathcal{T}_0$, and since $\varphi \vdash_{\mathsf{fde}} \psi \Rightarrow \varphi \vdash_{\mathsf{sm}} \psi$, also $\psi \notin \mathcal{T}_0$. Either ${\sim}\varphi \in \mathcal{T}_0$ or ${\sim}\varphi \notin \mathcal{T}_0$. If the latter is the case, then enumerate all the sentences of the language: $\chi_1, \chi_2, \ldots$, and build up a series of theories by defining for every $\mathcal{T}_n$ the next theory $\mathcal{T}_{n+1}$ as follows: (1) if ${\sim}\varphi \vee \psi \notin \mathcal{T}_n + \varphi_n$, then $\mathcal{T}_{n+1} = \mathcal{T}_n + \varphi_n$; (2) $\mathcal{T}_{n+1} = \mathcal{T}_n$, otherwise. The required theory $\mathcal{T}$ is then defined as the union of all the $\mathcal{T}_n$'s. $\mathcal{T}$ is a theory containing φ that is maximal with respect to the property of not containing ${\sim}\varphi \vee \psi$. By the usual argument one can show that $\mathcal{T}$ is prime. By using (di_1) and (dco) we also get that ${\sim}\varphi \notin \mathcal{T}$ and $\psi \notin \mathcal{T}$.

Assume ${\sim}\varphi \in \mathcal{T}_0$; then $\varphi \vdash_{\mathsf{fde}} {\sim}\varphi$. Since $\varphi \vdash_{\mathsf{fde}} \varphi$, by (ci) we have $\varphi \vdash_{\mathsf{fde}} \varphi \wedge {\sim}\varphi$. Hence, $\varphi \vdash_{\mathsf{sm}} \varphi \wedge {\sim}\varphi$, and by (saf) it turns out that φ is such that $\varphi \vdash_{\mathsf{sm}} \psi \vee {\sim}\psi$. In this case, consider theory $\mathcal{T}'_0 = \{{\sim}\varphi' : \varphi' \vdash_{\mathsf{fde}} \psi\}$. We have ${\sim}\psi \in \mathcal{T}'_0$, and ${\sim}\varphi \notin \mathcal{T}'_0$. Moreover, $\psi \notin \mathcal{T}'_0$. Indeed, assume $\psi \in \mathcal{T}'_0$; then ${\sim}\psi \vdash_{\mathsf{fde}} \psi$. Using $\psi \vdash_{\mathsf{fde}} \psi$, by (de) we get $\psi \vee {\sim}\psi \vdash_{\mathsf{fde}} \psi$, and hence $\psi \vee {\sim}\psi \vdash_{\mathsf{sm}} \psi$. By transitivity of $\vdash_{\mathsf{sm}}$ we get to $\varphi \vdash_{\mathsf{sm}} \psi$, contrary to the assumption of the lemma. The required theory $\mathcal{T}$ can be constructed as above. ◁

Lemma 28. *Let $\varphi \nvdash_{\mathsf{rm}^\wedge} \psi$. Then there is a prime FDE-theory $\mathcal{T}$, such that $\varphi \in \mathcal{T}$, ${\sim}\varphi \notin \mathcal{T}$ and $\psi \notin \mathcal{T}$, or there is a consistent prime FDE-theory $\mathcal{T}$, such that ${\sim}\varphi \notin \mathcal{T}$ and ${\sim}\psi \in \mathcal{T}$.*

Proof. Again, take as the starting theory $\mathcal{T}_0 = \{\psi' : \varphi \vdash_{\mathsf{fde}} \psi'\}$. We have $\varphi \in \mathcal{T}_0$, and since $\varphi \vdash_{\mathsf{fde}} \psi \Rightarrow \varphi \vdash_{\mathsf{rm}^\wedge} \psi$, also $\psi \notin \mathcal{T}_0$. Either ${\sim}\varphi \in \mathcal{T}_0$ or ${\sim}\varphi \notin \mathcal{T}_0$. If the latter is the case, then we continue as in Lemma 27 and are through.

Assume ${\sim}\varphi \in \mathcal{T}_0$; then $\varphi \vdash_{\mathsf{fde}} {\sim}\varphi$. Since $\varphi \vdash_{\mathsf{fde}} \varphi$, then by (ci) we have $\varphi \vdash_{\mathsf{fde}} \varphi \wedge {\sim}\varphi$. Now, consider theory $\mathcal{T}'_0 = \{{\sim}\varphi' : \varphi' \vdash_{\mathsf{fde}} \psi\}$. We have ${\sim}\psi \in \mathcal{T}'_0$, and ${\sim}\varphi \notin \mathcal{T}'_0$. Starting from $\mathcal{T}'_0$ we build up a series of theories by defining for every $\mathcal{T}_n$ the next theory $\mathcal{T}_{n+1}$ as follows: (1) if ${\sim}\varphi \notin \mathcal{T}_n + \varphi_n$, then $\mathcal{T}_{n+1} = \mathcal{T}_n + \varphi_n$; (2) $\mathcal{T}_{n+1} = \mathcal{T}_n$, otherwise. The required theory $\mathcal{T}$ can be defined as the union of all the $\mathcal{T}_n$, which is a maximal theory containing ${\sim}\psi$ with respect to the property of not having ${\sim}\varphi$. By the usual argument one can show that $\mathcal{T}$ is prime. Moreover, $\mathcal{T}$ is consistent. Indeed, assume it is not. Then, there is a formula χ, such that $\chi \in \mathcal{T}$, and ${\sim}\chi \in \mathcal{T}$. Hence, $\chi \vdash_{\mathsf{fde}} \psi$, and ${\sim}\chi \vdash_{\mathsf{fde}} \psi$. By (de) we obtain $\chi \vee {\sim}\chi \vdash_{\mathsf{fde}} \psi$. Recall that $\varphi \vdash_{\mathsf{fde}} \varphi \wedge {\sim}\varphi$. By (saf) we get to $\varphi \vdash_{\mathsf{fde}} \psi$. Then, $\varphi \vdash_{\mathsf{rm}^\wedge} \psi$, contrary to the assumption of the lemma. ◁

Lemma 29. *Let $\varphi \nvdash_{\mathsf{rm}^\vee} \psi$. Then there is a consistent prime FDE-theory $\mathcal{T}$, such that $\varphi \in \mathcal{T}$ and $\psi \notin \mathcal{T}$, or there is a prime FDE-theory $\mathcal{T}$, such that ${\sim}\varphi \notin \mathcal{T}$, ${\sim}\psi \in \mathcal{T}$ and $\psi \notin \mathcal{T}$.*

Proof. The lemma is proved dually to Lemma 28. ◁

And finally, we have the completeness theorem.

Theorem 30. *Let* s *be* sm, $\mathsf{rm}^\wedge$ *or* $\mathsf{rm}^\vee$. *Then for any* $\varphi, \psi \in \mathcal{L}_{\{\wedge,\vee,\sim\}}$, $\varphi \vDash_{\mathsf{s}} \psi \Rightarrow \varphi \vdash_{\mathsf{s}} \psi$.

Proof. 1. Let $\varphi \not\models_{\mathsf{sm}} \psi$. By Lemma 27, there is a prime FDE-theory $\mathcal{T}$, such that $\varphi \in \mathcal{T}$, $\sim\varphi \notin \mathcal{T}$, and $\psi \notin \mathcal{T}$, or there is a prime FDE-theory $\mathcal{T}$, such that $\sim\varphi \notin \mathcal{T}$, $\sim\psi \in \mathcal{T}$, and $\psi \notin \mathcal{T}$. Consider the canonical valuation defined in Lemma 25. We have $t \in v_\tau(\varphi)$, $f \notin v_\tau(\varphi)$, and $t \notin v_\tau(\psi)$; or $f \in v_\tau(\psi)$, $t \notin v_\tau(\psi)$, and $f \notin v_\tau(\varphi)$. That is, $v_\tau(\varphi) = T$ and $v_\tau(\psi) \neq T$, or $v_\tau(\psi) = F$ and $v_\tau(\varphi) \neq F$. Hence, $\varphi \not\models_{\mathsf{sm}} \psi$.

2. Let $\varphi \not\models_{\mathsf{rm}^\wedge} \psi$. Then, by Lemma 28, there is a prime FDE-theory $\mathcal{T}$, such that $\varphi \in \mathcal{T}$, $\sim\varphi \notin \mathcal{T}$ and $\psi \notin \mathcal{T}$, or there is a consistent prime FDE-theory $\mathcal{T}$, such that $\sim\varphi \notin \mathcal{T}$ and $\sim\psi \in \mathcal{T}$. Using the canonical valuation v_τ, we get $v_\tau(\varphi) = T$ and $v_\tau(\psi) \neq T$, or $f \in v_\tau(\psi)$ and $f \notin v_\tau(\varphi)$. Consistency of the theory in the second case ensures that v_τ is in fact v^3. Hence, $\varphi \not\models_{\mathsf{rm}^\wedge} \psi$.

3. The proof for $\varphi \not\models_{\mathsf{rm}^\vee} \psi$ is analogous. ◁

7. Safety, **R**-Mingle and Łukasiewicz's Logic

In this section, I will consider the issue of a proper characterization of $\mathbf{RM}_{\mathrm{S}}$, and in particular, a disputability of its association with some kind of "Kleene's logic," as, e.g., in [54; 1; 52]. As observed in Section 1, the latter connection arose from consideration of certain algebraic structure which is sometimes named after Kleene. In what follows, I will argue in favor of other logics as more suitable companions for $\mathbf{RM}_{\mathrm{S}}$, and also cast some doubt on the supposed connection of the algebraic structure in question with Kleene.

To this effect, I will give a proof of the fact mentioned above, namely, that $\mathbf{RM}_{\mathrm{S}}$ (and hence $\mathbf{RM}_{\mathsf{fde}}$) is indeed the first-degree entailment fragment of both **R**-Mingle (**RM**) and Łukasiewicz's three-valued logic **Ł3**. The proof will be essentially semantical. For axiomatic formulations of **RM** and **Ł3** in the language $\mathcal{L}_{\{\rightarrow,\wedge,\vee,\sim\}}$, consider the following list of axioms and rules of inference:

A1. $\varphi \rightarrow \varphi$
A2. $(\varphi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\varphi \rightarrow \chi))$
A3. $\varphi \rightarrow ((\varphi \rightarrow \psi) \rightarrow \psi)$
A4. $(\varphi \rightarrow (\varphi \rightarrow \psi)) \rightarrow (\varphi \rightarrow \psi)$
A5. $\varphi \rightarrow (\varphi \rightarrow \varphi)$
A6. $\varphi \rightarrow (\psi \rightarrow \varphi)$
A7. $(\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \chi))$
A8. $((\varphi \rightarrow \chi) \rightarrow \psi) \rightarrow (((\psi \rightarrow \varphi) \rightarrow \chi) \rightarrow \chi)$
A9. $(\varphi \wedge \psi) \rightarrow \varphi$ A10. $(\varphi \wedge \psi) \rightarrow \psi$
A11. $((\varphi \rightarrow \psi) \wedge (\varphi \rightarrow \chi)) \rightarrow (\varphi \rightarrow (\psi \wedge \chi))$
A12. $\varphi \rightarrow (\varphi \vee \psi)$ A13. $\psi \rightarrow (\varphi \vee \psi)$
A14. $((\varphi \rightarrow \chi) \wedge (\psi \rightarrow \chi)) \rightarrow ((\varphi \vee \psi) \rightarrow \chi)$
A15. $(\varphi \wedge (\psi \vee \chi)) \rightarrow ((\varphi \wedge \psi) \vee \chi)$
A16. $(\varphi \rightarrow \sim\psi) \rightarrow (\psi \rightarrow \sim\varphi)$
A17. $\sim\sim\varphi \rightarrow \varphi$ A18. $\varphi \rightarrow \sim\sim\varphi$
A19. $(\varphi \wedge \sim\varphi) \rightarrow (\psi \vee \sim\psi)$
R1. $\varphi \rightarrow \psi, \varphi \Rightarrow \psi$
R2. $\varphi, \psi \Rightarrow \varphi \wedge \psi$
R3. $\varphi \rightarrow \psi \Rightarrow \sim\psi \rightarrow \sim\varphi$

System **RM** is determined by A1–A5, A9–A17, R1, R2, see, e.g., [3, p. 341] and [4, §R], and system **Ł3** is determined by A6–A14, A17–A19, R1, R3, see Iturrioz [40, p. 618].

First consider semantics for **RM**. Dunn in [29] constructs a Kripke-style semantics for **R**-Mingle using a binary accessibility relation. Dunn's construction is explicitly based on the valuation $v^{3'}$ from footnote 10, but for the sake of uniformity with the approach adopted in the present paper, I will modify here that semantics with respect to valuation v^3.

Namely, define an **RM**-*model* as a triple $\langle W, R, v^3\rangle$, where W is a set, R is a reflexive, anti-symmetric, transitive, connected relation on W, and valuation v^3 is now relativized with respect to W, being thus defined as a map from $\mathcal{V}ar \times W$ into $\{T, F, N\}$, subject to the following *hereditary condition* (for any $p \in \mathcal{V}ar$, and $\alpha, \beta \in W$).

$$R\alpha\beta \Rightarrow v^3(p,\alpha) \subseteq v^3(p,\beta).$$

The valuation v^3 is extended to compound formulas as follows:

Definition 31.

1. $t \in v^3(\varphi \wedge \psi, \alpha) \;\Leftrightarrow\; t \in v^3(\varphi, \alpha)$ and $t \in v^3(\psi, \alpha)$,
   $f \in v^3(\varphi \wedge \psi, \alpha) \;\Leftrightarrow\; f \in v^3(\varphi, \alpha)$ or $f \in v^3(\psi, \alpha)$;
2. $t \in v^3(\varphi \vee \psi, \alpha) \;\Leftrightarrow\; t \in v^3(\varphi, \alpha)$ or $t \in v^3(\psi, \alpha)$,
   $f \in v^3(\varphi \vee \psi, \alpha) \;\Leftrightarrow\; f \in v^3(\varphi, \alpha)$ and $f \in v^3(\psi, \alpha)$;
3. $t \in v^3(\varphi \to \psi, \alpha) \;\Leftrightarrow\; \forall\beta\big[R\alpha\beta \Rightarrow [(t \in v^3(\varphi,\beta) \Rightarrow t \in v^3(\psi,\beta))$ and $(f \in v^3(\psi,\beta) \Rightarrow f \in v^3(\varphi,\beta))]\big]$,
   $f \in v^3(\varphi \to \psi, \alpha) \;\Leftrightarrow\; \exists\beta\big[R\alpha\beta$ and $[(t \in v^3(\varphi,\beta)$ and $f \in v^3(\psi,\beta))$ or $(f \in v^3(\psi,\beta)$ and $t \in v^3(\varphi,\beta))]\big]$;
4. $t \in v^3(\sim\varphi, \alpha) \;\Leftrightarrow\; f \in v^3(\varphi, \alpha)$, $\qquad f \in v^3(\sim\varphi, \alpha) \;\Leftrightarrow\; t \in v^3(\varphi, \alpha)$.

Let $\vDash_{\mathbf{RM}} \varphi$ mean that the formula φ is valid in the logic **RM**, and let $\vdash_{\mathbf{RM}} \varphi$ mean that φ is derivable in **RM**. One can define the notion of **RM**-*validity* as follows.

Definition 32. $\vDash_{RM} \varphi$ iff for any valuation v^3 in every RM-model $\langle W, R, v^3\rangle$, we have that $v^3(\varphi, \alpha) = T$.

And we have the following soundness and completeness theorem, the proof of which can be extracted from [29] *mutatis mutandis*.

Theorem 33. *For any* $\varphi \in \mathcal{L}_{\{\to,\wedge,\vee,\sim\}}$, $\vdash_{\mathbf{RM}} \varphi \;\Leftrightarrow\; \vDash_{RM} \varphi$.

Remark 34. The completeness proof of **RM** is given in [29] by a canonical model construction, where an **RM**-model is defined on the base of prime **RM**-theories. The canonical valuation v^c is then relativized with respect to the theories in a canonical model, so that for a propositional variable p, and a theory $\mathcal{T}$, $t \in v^c(p,\mathcal{T}) \Leftrightarrow p \in \mathcal{T}$, and $f \in v^c(p,\mathcal{T}) \Leftrightarrow \sim p \in \mathcal{T}$. It can be shown that the canonical valuation so defined can be extended to compound formulas by Definition 31.

To obtain semantics for **Ł3** define a matrix $\mathbf{Ł}_3 = \langle\{T, N, F\}, T, v^3\rangle$, where $\{T, N, F\}$ is as in Section 6, T is the designated element, and v^3 is again a map $\mathcal{V}ar \mapsto \{T, F, N\}$ extended to compound formulas by means of the following definitions for each connective from the language $\mathcal{L}_{\{\to,\wedge,\vee,\sim\}}$:

$v^3_{\to}$	T	N	F
T	T	N	F
N	T	T	N
F	T	T	T

$v^3_{\wedge}$	T	N	F
T	T	N	F
N	N	N	F
F	F	F	F

$v^3_{\vee}$	T	N	F
T	T	T	T
N	T	N	N
F	T	N	F

	$v^3_{\sim}$
T	F
N	N
F	T

Ł3-validity ($\models_{\textbf{Ł3}}$) can be defined as usual.

Definition 35. $\models_{\textbf{Ł3}} \varphi$ iff for any v^3 in valuation system $\textbf{Ł}_3$, $v^3(\varphi) = T$.

Ł3 is sound and complete with respect to this semantics:

Theorem 36. *For any* $\varphi \in \mathcal{L}_{\{\to,\wedge,\vee,\sim\}}$, $\vdash_{\textbf{Ł3}} \varphi \Leftrightarrow \models_{\textbf{Ł3}} \varphi$.

We are now in a position to establish the relationships between $\textbf{RM}_{\text{S}}$ on the one hand, and **RM** and **Ł3** on the other hand (cf. [34, Theorem 12]).

Lemma 37. *For any* $\varphi, \psi \in \mathcal{L}_{\{\wedge,\vee,\sim\}}$,
(1) $\varphi \to \psi$ *is provable in* **RM** *iff* $\varphi \vdash \psi$ *is provable in* $\textbf{RM}_{\text{S}}$*;*
(2) $\varphi \to \psi$ *is provable in* **Ł3** *iff* $\varphi \vdash \psi$ *is provable in* $\textbf{RM}_{\text{S}}$.

Proof. The direction from right to left is easy to establish by demonstrating that for any rule of $\textbf{RM}_{\text{S}}$ of the form $\varphi \vdash \psi$ the corresponding implication $\varphi \to \psi$ is provable both in **RM** and in **Ł3**, which is a routine exercise.

Moving in the opposite direction, assume that $\varphi \vdash \psi$ is *not* derivable in $\textbf{RM}_{\text{S}}$. Then:

(1) By Lemma 24, there is a consistent prime **RM**-theory $\mathcal{T}'$, such that $\varphi \in \mathcal{T}'$ and $\psi \notin \mathcal{T}'$, or there is a consistent prime **RM**-theory $\mathcal{T}''$, such that $\sim\varphi \notin \mathcal{T}''$ and $\sim\psi \in \mathcal{T}''$. Consider a canonical **RM**-model $\langle W^c, R^c, v^c\rangle$, where W^c is a set of consistent prime **RM**-theories, such that $\mathcal{T}', \mathcal{T}'' \in W^c$; $R^c\alpha\beta \Leftrightarrow \alpha \subseteq \beta$; and v^c is defined as in the remark above. In this **RM**-model, we thus have $\exists\alpha \in W^c(t \in v^c(\varphi,\alpha)$ and $t \notin v^c(\psi,\alpha))$, or $\exists\beta \in W^c(f \in v^c(\psi,\beta)$ and $f \notin v^c(\varphi,\beta))$. In both cases, we have $\not\models_{\textbf{RM}} \varphi \to \psi$, and, since **RM** is sound, $\varphi \to \psi$ is not provable in **RM**.

(2) By Theorem 18, $\varphi \not\models_{\mathsf{rm}} \psi$. Hence, $\exists v^3(t \in v^3(\varphi)$ and $t \notin v^3(\psi))$, or $\exists v^3(f \in v^3(\psi)$ and $f \notin v^3(\varphi))$. A simple inspection of the definition of $v^3_{\to}$ in the matrix $\textbf{Ł}_3$ shows that in both cases $t \notin v^3(\varphi \to \psi)$. Thus, $\not\models_{\textbf{Ł3}} \varphi \to \psi$, and so $\nvdash_{\textbf{Ł3}} \varphi \to \psi$. ◁

Taking into account the deductive equivalence between $\textbf{RM}_{\text{S}}$ and $\textbf{RM}_{\mathsf{fde}}$, this proof provides also the proof of Lemma 19.

Now, returning to the problem of a suitable characterization of the logic with Safety as a distinctive principle, consider algebraic structures usually employed on this issue. The basic structure here is De Morgan algebra.[11] It can be defined as a structure $\langle A, \cap, \cup, -, 1\rangle$, where $\langle A, \cap, \cup, 1\rangle$ is a distributive lattice with greatest element 1, and $-$ is a unary operation on A satisfying the following conditions:

(1) $$--x = x,$$

(2) $$-(x \cup y) = -x \cap -y.$$

[11] Béziau in [16, p. 280] explains: "The idea of De Morgan algebra can be traced back to Moisil's paper [47]. They were later on studied in Poland and called *quasi-boolean algebras* by Rasiowa [17]. They also have been called *distributive i-lattices* by Kalman [42]."

Brignole and Monteiro in [22] came up with a new name "Kleene algebra" for a structure obtained by equipping a De Morgan algebra with an additional condition, which is an algebraic counterpart of (saf):[12]

(3) $$x \cap -x \leq y \cup -y.$$

They seem to justify this name by an observation that (saf) is verified in Kleene's (strong) three-valued logic, see [22, p. 4]. Kaarli and Pixley propose in [41, p. 296] another definition of Kleene algebra by taking the equation

(4) $$(x \cap -x) \cup (y \cup -y) = y \cup -y$$

instead of (3). Clearly, (3) and (4) are interderivable. It is observed that the variety of Kleene algebras so defined is generated by a special structure $K_3 = \langle \{0, a, 1\}, \cap, \cup, -, 1\rangle$ (notation adjusted), with $0 < a < 1$, and $-a = a$. The label K_3 apparently suggests the association of this structure with Kleene's logic. It is, however, noteworthy that the structure in question is not uniquely Kleenean, and may well serve as an algebraic background for other three-valued logics.

Most importantly, neither (3) nor (4) is characteristic for Kleene's logic. Indeed, although (saf) is a valid consequence of Kleene's logic, it is so only as a substitutional case of a more general principle (efq) provable there. Moreover, the implicational version of Safety is *not* a theorem of Kleene's logic, since the latter has no theorems at all. Therefore, Brignole and Monteiro's justification of the name "Kleene algebra" seems not very persuasive.

At the same time, an easy check shows that implicational version of Safety is valid in Łukasiewicz's three-valued logic, and (saf) is valid there by itself, without (veq) or (efq) being valid. Lemma 37 states essentially that (saf) is indeed characteristic for the non-implicational fragments of both **R**-Mingle and Łukasiewicz's three-valued logic. Thus, it could be more appropriate to associate the algebraic structure in question with the name of Łukasiewicz rather than Kleene.

Itturioz in [40] defines a three-valued Łukasiewicz algebra as a structure $\langle A, \cap, \cup, \Rightarrow, -, 1\rangle$, where $\langle A, \cap, \cup, \Rightarrow, 1\rangle$ is a relatively pseudo-complemented lattice with the following additional condition for $\Rightarrow$:[13]

(5) $$((x \Rightarrow z) \Rightarrow y) \Rightarrow (((y \Rightarrow x) \Rightarrow y) \Rightarrow y) = 1,$$

and the unary operation $-$ is subject to the conditions (1) and (2) above, as well as the following additional condition:

(6) $$(x \cap -x) \cap (y \cup -y) = x \cap -x.$$

Of course, (3), (4) and (6) are all equivalent. Furthermore, a Łukasiewicz algebra so defined is explicitly formulated in the signature $\Omega = \langle \cap, \cup, \Rightarrow, -\rangle$. Now, if one removes from this signature the operation of relative pseudo-complement together with the corresponding conditions, one obtains exactly the structure, which Brignole and

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[12] As already said in Section 1, Kalman called such structure a "normal i-lattice."

[13] A relatively pseudo-complemented lattice is a structure $\langle A, \cap, \cup, \Rightarrow, 1\rangle$, where $\langle A, \cap, \cup\rangle$ is a lattice, and the following condition holds: $x \cap y \leq z$ iff $x \leq y \Rightarrow z$. The element $x \Rightarrow y$ is called the pseudo-complement of x relative to y, see [53, pp. 52–53].

Monteiro called "Kleene algebra," but which might be better called "Kalman algebra" or "quasi-Łukasiewicz algebra."

The corresponding consequence system is the first-degree entailment fragment of Łukasiewicz's three-valued logic, which is coincident with the first-degree entailment fragment of the logic **R**-Mingle, and which is a *subsystem* of Kleene's strong three-valued logic.

8. Concluding Remarks

Remarkably, the first individual full-fledged paper [27] published by J. Michael Dunn (if not to take into account a short note [26]) was devoted to **R**-Mingle. The development of this logical system at the start of Dunn's rich and fruitful scientific career, which spanned more than half a century, attested to his exceptional talent and abilities in the field of philosophical logic. It is also rather symbolic that the last (individual) paper Dunn apparently was working on, which has been posthumously published in [33] was also dealing with this logical system. This clearly reaffirms the importance of the mingle logics and the mingle principle in modern logical investigations.

In this paper, I argued for the applicability of the mingle principle as a kind of safety-lock that helps to avoid the most disastrous consequences of the paradoxes of relevance even in the presence of *some* irrelevant inferences. Furthermore, I have concentrated on a specific implementation of the mingle principle on the first-degree entailment level, dubbed "Safety." As it turns out, the first-degree entailment framework allows for a more subtle distinction between four main versions of this principle, which form a four-element diamond-shaped lattice of what can be called "FDE-based mingle logics," with infinitely many intermediate systems in between. The corner systems of that lattice have a very natural and uniform semantics in terms of the forward truth preservation and backward falsity preservation. It would be interesting to extend the proposed semantic framework to the whole infinity of systems from our diamond.

Moreover, it was also observed that the first-degree fragment of **RM** is devoid of a rather problematic irrelevant property (CP), which links together any propositions of our language. In the first-degree entailment context this property is inexpressible on the level of the object language; moreover, its meta-language formulation does not hold in $\mathbf{RM}_{\mathrm{S}}$ either. That is, there are formulas φ and ψ in the language $\mathcal{L}_{\{\wedge,\vee,\sim\}}$, such that neither $\varphi \vdash \psi$ nor $\psi \vdash \varphi$ is provable in $\mathbf{RM}_{\mathrm{S}}$. This observation suggests a promising direction of future work — to consider the system $\mathbf{RS} = \mathbf{R} + \text{Safety}$.

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