Tutorial on Truth Values

Session 2: Logics of generalized truth values

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In (S&W 2005) the axiomatizations of truth and falsity entailment were extended to the languages $\mathcal{L}_t \cup \{\sim_f\}$ and $\mathcal{L}_f \cup \{\sim_t\}$. The problem of axiomatizing $\models_t$ and $\models_f$ in the full language $\mathcal{L}_{tf}$, however, was left open.

In (W&S 2008a) it has been suggested to extend the language $\mathcal{L}_{tf}$ by implication connectives, namely by a truth implication $\rightarrow_t$ interpreted as the residuum of the truth order with respect to truth conjunction. Falsity implication $\rightarrow_f$ can then be interpreted as the residuum of the falsity order with respect to falsity conjunction.

Definition

The languages $\mathcal{L}_{tf}^{\rightarrow_t}$, $\mathcal{L}_{tf}^{\rightarrow_f}$, and $\mathcal{L}_{tf}^{*}$ are defined as $\mathcal{L}_{tf} \cup \{\rightarrow_t\}$, $\mathcal{L}_{tf} \cup \{\rightarrow_f\}$, and $\mathcal{L}_{tf} \cup \{\rightarrow_t, \rightarrow_f\}$, respectively.

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Sergei Odintsov (2009) has presented an axiomatization of truth entailment and falsity entailment in $SIXTEEN_3$ in the enriched languages $\mathcal{L}_{tf}^{\rightarrow t}$, $\mathcal{L}_{tf}^{\rightarrow f}$, and $\mathcal{L}_{tf}^{*}$.

A central idea for defining these axiomatizations is to present $SIXTEEN_3$ as a four-component twist-structure over the two-element Boolean algebra. This matrix presentation of the algebraic operations of $SIXTEEN_3$ will give rise to the definition of the cut-free sequent calculi for truth and falsity entailment to be presented.

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Odintsov (2009) starts with an auxiliary, first-degree axiomatic proof systems $\vdash_{base}$ for the language $\mathcal{L}_{tf}$.

Next, he defines two extensions $\vdash_T$ and $\vdash_B$ of the basic system $\vdash_{base}$ and shows that $\models_t = \vdash_T \cap \vdash_B$.

Similarly, Odintsov defines first-degree proof systems $\vdash^f_{base}$, $\vdash^f_T$, and $\vdash^f_B$ for the language $\mathcal{L}_{tf}$, such that $\models_f = \vdash^f_T \cap \vdash^f_B$.

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The language $\mathcal{L}_{tf}$ seems to be too weak to axiomatize the intersection of $\vdash_T$ and $\vdash_B$.

Since $\vdash_T$ and $\vdash_B$, respectively, $\vdash_T^f$ and $\vdash_B^f$ do not have modus ponens as their only rule of inference, it is not clear how to axiomatize their intersections. Miura's Theorem on the axiomatization of intersections of Hilbert-style proof systems for superintuitionistic logics, uses the Deduction Theorem for intuitionistic implication and applies to logics with axiomatizations comprising modus ponens as the only inference rule.

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To apply Miura's Theorem, Odintsov considers the implicational languages $\mathcal{L}_{tf}^{\rightarrow t}$, $\mathcal{L}_{tf}^{\rightarrow f}$, and $\mathcal{L}_{tf}^{*}$.

Moreover, Odintsov introduces a simple and elegant matrix presentation of the algebraic operations in $SIXTEEN_3$. $SIXTEEN_3$ is thereby presented as a twist structure because its operations "are defined componentwise on several components and are "twisted" in some way on the others".

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Every element x of **16** is a subset of $\mathcal{P}(\mathbf{2})$; therefore it can be represented as a 2×2-matrix of values of characteristic functions:

$$\begin{vmatrix} n & f \\ t & b \end{vmatrix}$$

where each element of the matrix is an element of $\{0, 1\}$ (understood as the set of classical truth values) and the following equivalences hold:

$$n = 1 \text{ iff } \mathbf{N} \in x; \quad f = 1 \text{ iff } \mathbf{F} \in x;$$
$$t = 1 \text{ iff } \mathbf{T} \in x; \quad b = 1 \text{ iff } \mathbf{B} \in x.$$

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Proposition (Odintsov)

Let $\wedge$ and $\vee$ be the classical truth functions of conjunction and disjunction, let $x, y \in \mathbf{16}$, let $x = \begin{vmatrix} n & f \\ t & b \end{vmatrix}$ *and let* $y = \begin{vmatrix} n' & f' \\ t' & b' \end{vmatrix}$.

Then the following equations hold:

$$\begin{vmatrix} n & f \\ t & b \end{vmatrix} \sqcap_t \begin{vmatrix} n' & f' \\ t' & b' \end{vmatrix} = \begin{vmatrix} n \vee n' & f \vee f' \\ t \wedge t' & b \wedge b' \end{vmatrix}$$

$$\begin{vmatrix} n & f \\ t & b \end{vmatrix} \sqcup_t \begin{vmatrix} n' & f' \\ t' & b' \end{vmatrix} = \begin{vmatrix} n \wedge n' & f \wedge f' \\ t \vee t' & b \vee b' \end{vmatrix}$$

$$-_t \begin{vmatrix} n & f \\ t & b \end{vmatrix} = \begin{vmatrix} t & b \\ n & f \end{vmatrix}.$$

Proposition (continued)

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The operations $\sqsupset_t$ and $\sqsupset_f$ on **16** are defined as follows:

$$\left|\begin{array}{cc} n & f \\ t & b \end{array}\right| \sqsupset_t \left|\begin{array}{cc} n' & f' \\ t' & b' \end{array}\right| := \left|\begin{array}{cc} \neg n \wedge n' & \neg f \wedge f' \\ t \to t' & b \to b' \end{array}\right|$$

$$\left|\begin{array}{cc} n & f \\ t & b \end{array}\right| \sqsupset_f \left|\begin{array}{cc} n' & f' \\ t' & b' \end{array}\right| := \left|\begin{array}{cc} n \to n' & \neg f \wedge f' \\ t \to t' & \neg b \wedge b' \end{array}\right|$$

where $\to$ and $\neg$ are the truth functions of Boolean implication and Boolean negation.

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The definition of valuation functions is extended by the clauses

$$v(A \to_t B) = v(A) \sqsupset_t v(B), \quad v(A \to_f B) = v(A) \sqsupset_f v(B).$$

The two biconditionals $A \leftrightarrow_t B$ and $A \leftrightarrow_f B$ are defined as $(A \to_t B) \wedge_t (B \to_t A)$ and $(A \to_f B) \wedge_f (B \to_f A)$, respectively, and the classical negation $\neg A$ of A can be defined in $\mathcal{L}_{tf}^{\to_t}$ as $A \to_t \sim_t (p \to_t p)$ and in $\mathcal{L}_{tf}^{\to_f}$ as $A \to_f \sim_f (p \to_f p)$, where p is some fixed propositional variable.

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Proposition (Odintsov)

For every $x, y, z \in \mathbf{16}$,

1. $x \leq_t y \sqsupset_t z$ *iff* $x \sqcap_t y \leq_t z$ *and*
2. $x \leq_f y \sqsupset_f z$ *iff* $x \sqcap_f y \leq_f z$.

Corollary

1. *For every* $x, y \in \mathbf{16}$, $x \leq_t y$ *iff* $x \sqsupset_t y = \mathbf{TB}$.
2. *For every formula* $A, B \in \mathcal{L}_{tf}^{\rightarrow t}$, $A \models_t B$ *iff* $\forall v(v(A \rightarrow_t B) = \mathbf{TB})$.
3. *For every* $x, y \in \mathbf{16}$, $x \leq_f y$ *iff* $x \sqsupset_f y = \mathbf{NT}$.
4. *For every formula* $A, B \in \mathcal{L}_{tf}^{\rightarrow f}$, $A \models_f B$ *iff* $\forall v(v(A \rightarrow_f B) = \mathbf{NT})$.

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Odintsov axiomatizes the logic $L^t = \{A \mid \forall v\,(v(A) = \mathbf{TB}),\ A \in \mathcal{L}_{tf}^{\rightarrow t}\}$ as the intersection of two logics L_T and L_B presented as Hilbert-style proof systems with modus ponens as the only rule of inference.

Similarly, he axiomatizes the logic $L^{t*} = \{A \mid \forall v\,(v(A) = \mathbf{TB}),\ A \in \mathcal{L}_{tf}^{*}\}$ as the intersection of two logics L_T^* and L_B^* presented as axiom systems with modus ponens as the only inference rule.
Axiomatizations for the semantically defined logics $L^f = \{A \mid \forall v\,(v(A) = \mathbf{NT}),\ A \in \mathcal{L}_{tf}^{\rightarrow f}\}$ and $L^{f*} = \{A \mid \forall v\,(v(A) = \mathbf{NT}),\ A \in \mathcal{L}_{tf}^{*}\}$ can be obtained analogously.

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We shall here focus on the systems explicitly treated by Odintsov, that is, L^t and L^{t*}.

Definition

The logic L_{base} in the language $\mathcal{L}_{tf}^{\to_t}$ is the least set of formulas closed under modus ponens for $\to_t$

$$\frac{A \quad A \to_t B}{B}$$

and containing the following axioms:

1. $A \to_t (B \to_t A)$
2. $(A \to_t (B \to_t C)) \to_t ((A \to_t B) \to_t (A \to_t C))$
3. $(A \wedge_t B) \to_t A$
4. $(A \wedge_t B) \to_t B$
5. $(A \to_t B) \to_t ((A \to_t C) \to_t (A \to_t (B \wedge_t C)))$
6. $A \to_t (A \vee_t B)$
7. $B \to_t (A \vee_t B)$

Definition (continued)

8. $(A \to_t C) \to_t ((B \to_t C) \to_t ((A \vee_t B) \to_t C))$
9. $A \vee_t (A \to_t B)$
10. $A \leftrightarrow_t \sim_t \sim_t A$
11. $A \leftrightarrow_t \sim_f \sim_f A$
12. $\sim_t \sim_f A \leftrightarrow_t \sim_f \sim_t A$
13. $\neg \sim_f A \leftrightarrow_t \sim_f \neg A$
14. $\neg \sim_t A \leftrightarrow_t \sim_t \neg A$
15. $\neg \sim_f \sim_t A \leftrightarrow_t \sim_f \sim_t \neg A$
16. $\sim_f A \leftrightarrow_t \sim_t \sim_f \sim_t A$
17. $\sim_t A \leftrightarrow_t \sim_f \sim_t \sim_f A$

Definition (continued)

18. $\sim_t (A \wedge_t B) \leftrightarrow_t (\sim_t A \vee_t \sim_t B)$
19. $\sim_t (A \vee_t B) \leftrightarrow_t (\sim_t A \wedge_t \sim_t B)$
20. $\sim_t (A \wedge_f B) \leftrightarrow_t (\sim_t A \wedge_f \sim_t B)$
21. $\sim_t (A \vee_f B) \leftrightarrow_t (\sim_t A \vee_f \sim_t B)$
22. $\sim_f (A \wedge_f B) \leftrightarrow_t (\sim_f A \vee_f \sim_f B)$
23. $\sim_f (A \vee_f B) \leftrightarrow_t (\sim_f A \wedge_f \sim_f B)$
24. $\sim_f (A \wedge_t B) \leftrightarrow_t (\sim_f A \wedge_t \sim_f B)$
25. $\sim_f (A \vee_t B) \leftrightarrow_t (\sim_f A \vee_t \sim_f B)$
26. $\sim_f \sim_t (A \wedge_t B) \leftrightarrow_t (\sim_f \sim_t A \vee_t \sim_f \sim_t B)$
27. $\sim_f \sim_t (A \vee_t B) \leftrightarrow_t (\sim_f \sim_t A \wedge_t \sim_f \sim_t B)$

Definition (continued)

28. $\sim_f\sim_t (A \wedge_f B) \leftrightarrow_t (\sim_f\sim_t A \vee_f \sim_f\sim_t B)$
29. $\sim_f\sim_t (A \vee_f B) \leftrightarrow_t (\sim_f\sim_t A \wedge_f \sim_f\sim_t B)$
30. $(A \rightarrow_t B) \leftrightarrow_t (\neg A \vee_t B)$
31. $\sim_t (A \rightarrow_t B) \leftrightarrow_t (\sim_t \neg A \wedge_t \sim_t B)$
32. $\sim_f (A \rightarrow_t B) \leftrightarrow_t (\sim_f A \rightarrow_t \sim_f B)$
33. $\sim_f\sim_t (A \rightarrow_t B) \leftrightarrow_t (\sim_f\sim_t \neg A \wedge_t \sim_f\sim_t B)$

Definition

The axiom systems L_T and L_B are defined as follows:

$$L_T := L_{base} \cup \{(A \wedge_f B) \leftrightarrow_t (A \wedge_t B), (A \vee_f B) \leftrightarrow_t (A \vee_t B)\},$$

$$L_B := L_{base} \cup \{(A \wedge_f B) \leftrightarrow_t (A \vee_t B), (A \vee_f B) \leftrightarrow_t (A \wedge_t B)\}.$$

Definition

The axiom system HL^t consists of L_{base} together with the following disjunction:

$$(((A \wedge_t B) \leftrightarrow_t (A \wedge_f B)) \wedge_t ((A \vee_t B) \leftrightarrow_t (A \vee_f B)))\vee_t$$
$$(((C \wedge_t D) \leftrightarrow_t (C \vee_f D)) \wedge_t ((C \vee_t D) \leftrightarrow_t (C \wedge_f D))).$$

Definition

The axiom systems L_T and L_B are defined as follows:

$$L_T := L_{base} \cup \{(A \wedge_f B) \leftrightarrow_t (A \wedge_t B), (A \vee_f B) \leftrightarrow_t (A \vee_t B)\},$$

$$L_B := L_{base} \cup \{(A \wedge_f B) \leftrightarrow_t (A \vee_t B), (A \vee_f B) \leftrightarrow_t (A \wedge_t B)\}.$$

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$$(((A \wedge_t B) \leftrightarrow_t (A \wedge_f B)) \wedge_t ((A \vee_t B) \leftrightarrow_t (A \vee_f B)))\vee_t$$
$$(((C \wedge_t D) \leftrightarrow_t (C \vee_f D)) \wedge_t ((C \vee_t D) \leftrightarrow_t (C \wedge_f D))).$$

Definition

The logic L^*_{base} is the least logic in the language $\mathcal{L}^*_{tf}$ closed under modus ponens for $\rightarrow_t$ and $\rightarrow_f$, comprising the axioms of L_{base}, and containing the following axioms for $\rightarrow_f$:

34. $(A \rightarrow_f B) \leftrightarrow_t (\neg A \vee_f B)$
35. $\sim_t (A \rightarrow_f B) \leftrightarrow_t (\sim_t A \rightarrow_f \sim_t B)$
36. $\sim_f (A \rightarrow_f B) \leftrightarrow_t (\sim_f \neg A \wedge_f \sim_f B)$
37. $\sim_f \sim_t (A \rightarrow_f B) \leftrightarrow_t (\sim_f \sim_t \neg A \wedge_f \sim_f \sim_t B)$.

Definition

The axiom systems L^*_T and L^*_B are defined as follows:

$$L^*_T := L^*_{base} \cup \{(A \wedge_t B) \leftrightarrow_t (A \wedge_f B), (A \vee_t B) \leftrightarrow_t (A \vee_f B), (A \rightarrow_f B) \leftrightarrow_t (A \rightarrow_t B)\},$$

$$L^*_B := L^*_{base} \cup \{(A \wedge_t B) \leftrightarrow_t (A \vee_f B), (A \vee_t B) \leftrightarrow_t (A \wedge_f B), (A \rightarrow_f B) \leftrightarrow_t (\neg A \wedge_t B)\}.$$

Definition

The axiom system HL^{t*} consists of L^*_{base} together with the following disjunction:

38.

$$((A \wedge_t B) \leftrightarrow_t (A \wedge_f B)) \wedge_t ((A \vee_t B) \leftrightarrow_t (A \vee_f B)) \wedge_t ((A \rightarrow_f B) \leftrightarrow_t (A \rightarrow_t B)) \vee_t$$
$$((C \wedge_t D) \leftrightarrow_t (C \vee_f D)) \wedge_t ((C \vee_t D) \leftrightarrow_t (C \wedge_f D)) \wedge_t ((C \rightarrow_f D) \leftrightarrow_t (\neg C \wedge_t D))).$$

Definition

The axiom systems L_T^* and L_B^* are defined as follows:

$$L_T^* := L_{base}^* \cup$$
$$\{(A \wedge_t B) \leftrightarrow_t (A \wedge_f B), (A \vee_t B) \leftrightarrow_t (A \vee_f B), (A \rightarrow_f B) \leftrightarrow_t (A \rightarrow_t B)\},$$

$$L_B^* := L_{base}^* \cup$$
$$\{(A \wedge_t B) \leftrightarrow_t (A \vee_f B), (A \vee_t B) \leftrightarrow_t (A \wedge_f B), (A \rightarrow_f B) \leftrightarrow_t (\neg A \wedge_t B)\}.$$

Definition

The axiom system HL^{t*} consists of L_{base}^* together with the following disjunction:

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$$(A \wedge_t B) \leftrightarrow_t (A \wedge_f B)) \wedge_t ((A \vee_t B) \leftrightarrow_t (A \vee_f B)) \wedge_t ((A \rightarrow_f B) \leftrightarrow_t (A \rightarrow_t B)))\vee_t$$
$$(C \wedge_t D) \leftrightarrow_t (C \vee_f D)) \wedge_t ((C \vee_t D) \leftrightarrow_t (C \wedge_f D)) \wedge_t ((C \rightarrow_f D) \leftrightarrow_t (\neg C \wedge_t D))).$$

Theorem (Odintsov)

Let A be a formula of $\mathcal{L}_{tf}^{\rightarrow t}$. Then $A \in L^t$ iff $HL^t \vdash A$.
Let A be a formula of $\mathcal{L}_{tf}^{*}$. Then $A \in L^{t*}$ iff $HL^{t*} \vdash A$.

We will consider sequent calculi for HL^t and HL^{t*} and other trilattice logics in $\mathcal{L}_{tf}^{\rightarrow t}$, $\mathcal{L}_{tf}^{\rightarrow f}$, and $\mathcal{L}_{tf}^{*}$.

Theorem (Odintsov)

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We will consider sequent calculi for HL^t and HL^{t*} and other trilattice logics in $\mathcal{L}_{tf}^{\rightarrow t}$, $\mathcal{L}_{tf}^{\rightarrow f}$, and $\mathcal{L}_{tf}^{*}$.

	axiom system	language	relation to $\models_t$ or $\models_f$	inference rules
(1)	$\vdash_{base}$	$\mathcal{L}_{tf}$	$\vdash_{base} \subseteq \models_t$	not just m. p.
	$\vdash_T$	$\mathcal{L}_{tf}$	$\models_t = \vdash_T \cap \vdash_F$	not just m. p.
	$\vdash_B$	$\mathcal{L}_{tf}$	$\models_t = \vdash_T \cap \vdash_F$	not just m. p.
(2)	$\vdash^f_{base}$	$\mathcal{L}_{tf}$	$\vdash^f_{base} \subseteq \models_f$	not just m. p.
	$\vdash^f_T$	$\mathcal{L}_{tf}$	$\models_f = \vdash^f_T \cap \vdash^f_F$	not just m. p.
	$\vdash^f_B$	$\mathcal{L}_{tf}$	$\models_f = \vdash^f_T \cap \vdash^f_F$	not just m. p.

Table: Axiom systems related to $SIXTEEN_3$.

	axiom system	language	relation to $\models_t$ or $\models_f$	inference rules
(3)	L_{base}	$\mathcal{L}_{tf}^{\rightarrow_t}$	$L_{base} \subseteq \models_t$	just m. p. for $\rightarrow_t$
	L_T	$\mathcal{L}_{tf}^{\rightarrow_t}$	$L^t = L_T \cap L_F$	just m. p. for $\rightarrow_t$
	L_B	$\mathcal{L}_{tf}^{\rightarrow_t}$	$L^t = L_T \cap L_F$	just m. p. for $\rightarrow_t$
(4)	L'_{base}	$\mathcal{L}_{tf}^{\rightarrow_t}$	$L'_{base} \subseteq \models_f$	just m. p. for $\rightarrow_t$
	L'_T	$\mathcal{L}_{tf}^{\rightarrow_t}$	$L' = L'_T \cap L'_F$	just m. p. for $\rightarrow_t$
	L'_B	$\mathcal{L}_{tf}^{\rightarrow_t}$	$L' = L'_T \cap L'_F$	just m. p. for $\rightarrow_t$
(5)	L^f_{base}	$\mathcal{L}_{tf}^{\rightarrow_f}$	$L^f_{base} \subseteq \models_f$	just m. p. for $\rightarrow_f$
	L^f_T	$\mathcal{L}_{tf}^{\rightarrow_f}$	$L^f = L^f_T \cap L^f_F$	just m. p. for $\rightarrow_f$
	L^f_B	$\mathcal{L}_{tf}^{\rightarrow_f}$	$L^f = L^f_T \cap L^f_F$	just m. p. for $\rightarrow_f$
(6)	$L^{f'}_{base}$	$\mathcal{L}_{tf}^{\rightarrow_f}$	$L^{f'}_{base} \subseteq \models_t$	just m. p. for $\rightarrow_f$
	$L^{f'}_T$	$\mathcal{L}_{tf}^{\rightarrow_f}$	$L^{f'} = L^{f'}_T \cap L^{f'}_F$	just m. p. for $\rightarrow_f$
	$L^{f'}_B$	$\mathcal{L}_{tf}^{\rightarrow_f}$	$L^{f'} = L^{f'}_T \cap L^{f'}_F$	just m. p. for $\rightarrow_f$

Table: Axiom systems related to $SIXTEEN_3$.

	axiom syst.	lang.	relation to $\models_t$ or $\models_f$	inference rules
(7)	L^{t*}_{base}	$\mathcal{L}^*_{tf}$	$L^{t*}_{base} \subseteq \models_t$	just m. p. for $\rightarrow_t$ and $\rightarrow_f$
	L^{t*}_T	$\mathcal{L}^*_{tf}$	$L^{t*} = L^{t*}_T \cap L^{t*}_F$	just m. p. for $\rightarrow_t$ and $\rightarrow_f$
	L^{t*}_B	$\mathcal{L}^*_{tf}$	$L^{t*} = L^{t*}_T \cap L^{t*}_F$	just m. p. for $\rightarrow_t$ and $\rightarrow_f$
(8)	L^{f*}_{base}	$\mathcal{L}^*_{tf}$	$L^{f*}_{base} \subseteq \models_f$	just m. p. for $\rightarrow_t$ and $\rightarrow_f$
	L^{f*}_T	$\mathcal{L}^*_{tf}$	$L^{f*} = L^{f*}_T \cap L^{f*}_F$	just m. p. for $\rightarrow_t$ and $\rightarrow_f$
	L^{*f}_B	$\mathcal{L}^*_{tf}$	$L^{*f} = L^{f*}_T \cap L^{f*}_F$	just m. p. for $\rightarrow_t$ and $\rightarrow_f$

Table: Axiom systems related to $SIXTEEN_3$.

Gentzen introduced sequents $\Delta \vdash \Gamma$, where Δ and Γ are finite sequences of formulas. Various structural inference rules not displaying any logical operations are assumed such that Δ and Γ may be taken to be finite sets and the intuitive understanding of $\Delta \vdash \Gamma$ is expressed by the formula $\bigwedge \Delta \to \bigvee \Gamma$.

If we translate into classical logic and $\Delta = \{A_1, \ldots, A_n\}$, then the translation of $\Delta \vdash \Gamma$ is equivalent with $\bigvee\{\neg A_1, \ldots, \neg A_n\} \vee \bigvee \Gamma$. Thus, $\Delta \vdash \Gamma$ is understood as a disjunction of *true* formulas (in Γ) and formulas which are *not true* (in Δ).

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Then for every logical operation there are rules for introducing the operation into the position of formulas that are true and the position of formulas that are not true.

This perspective on sequents can be generalized to n-valued logics. A sequent is a structure $\Delta_1 \mid \ldots \mid \Delta_n$ with the reading "some formula in Δ_1 has the value 1 or ... or some formula in Δ_n has the value n". A sequent calculus for an n-valued logic then has rules for introducing operations into each of the n positions of a sequent $\Delta_1 \mid \ldots \mid \Delta_n$.

It is known for a long time that there exists a systematic way to define such many-sided sequent systems for any finitely-valued logic defined by a logical matrix together with a set of designated values. $\models_t$ and $\models_f$ in $SIXTEEN_3$ are, however, not defined in this way. Nevertheless, a kind of many-sided sequents can be used to define sequent calculi for the trilattice logics L^t and L^{t*}.

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It is not necessary, however, to introduce 16-place sequents $\Delta_1 \mid \ldots \mid \Delta_{16}$. As we have seen, every value from **16** ($= \mathcal{P}(\mathbf{4})$) is determined by a set of *four* characteristic functions. A (generalized) sequent may therefore be defined as a structure $s =$

$$\Delta_1 \circ \Gamma_1 \mid \ldots \mid \Delta_4 \circ \Gamma_4.$$

where Δ_i and Γ_i ($i \in \{1, 2, 3, 4\}$) are finite sets of formulas.

Intuitive reading of s: "(The semantic value of some formula in Δ_1 does not contain **N** or the semantic value of some formula in Γ_1 contains **N**) and (the semantic value of some formula in Δ_2 does not contain **F** or the semantic value of some formula in Γ_2 contains **F**) and (the semantic value of some formula in Δ_3 does not contain **T** or the semantic value of some formula in Γ_3 contains **T**) and (the semantic value of some formula in Δ_4 does not contain **B** or the semantic value of some formula in Γ_4 contains **B**.)"

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We write A, if a set Δ_i or Γ_i is a singleton set $\{A\}$, we write A, Δ_i or Δ_i, A instead of $\{A\} \cup \Delta_i$ (and similarly for Γ_i), and we just leave an empty space, if a set Δ_i or Γ_i is empty.

Axiomatic sequents are sequents $\Delta_1 \circ \Gamma_1 \mid \Delta_2 \circ \Gamma_2 \mid \Delta_3 \circ \Gamma_3 \mid \Delta_4 \circ \Gamma_4$, where at least one $\Delta_i \circ \Gamma_i$ $(1 \leq i \leq 4)$ has the shape $\Delta, p \circ \Gamma, p$ for some atomic formula p. A formula A in Δ_i (Γ_i) is said to occur in position Δ_i (Γ_i), and the sequent $\circ \mid \circ \mid \circ \mid \circ$ is called the empty sequent.

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The idea is to define a calculus such that the (generalized) sequents

$$A\circ \mid \circ \mid \circ \mid \circ, \quad \circ \mid A\circ \mid \circ \mid \circ, \quad \circ \mid \circ \mid \circ A \mid \circ, \quad \circ \mid \circ \mid \circ \mid \circ A$$

are provable iff A receives the greatest element of $\leq_t$ under every valuation, i.e., iff it is interpreted as **TB** under every valuation.

An expression $\Delta_i : A, s$ or $\Gamma_i : A, s$ for each $i \in \{1, 2, 3, 4\}$ is used to denote the place-wise union of s and $\{A\}$ at position Δ_i or Γ_i, respectively, for example, $\Delta_3 : A, s$ means $\Delta_1 \circ \Gamma_1 \mid \Delta_2 \circ \Gamma_2 \mid A, \Delta_3 \circ \Gamma_3 \mid \Delta_4 \circ \Gamma_4$. By s, s' we shall denote the place-wise union of two sequents s and s'. Thus, if $s = \Delta_1 \circ \Gamma_1 \mid \ldots \mid \Delta_4 \circ \Gamma_4$ and $s' = \Delta'_1 \circ \Gamma'_1 \mid \ldots \mid \Delta'_4 \circ \Gamma'_4$, then $s, s' = \Delta_1 \cup \Delta'_1 \circ \Gamma_1 \cup \Gamma'_1 \mid \ldots \mid \Delta_4 \cup \Delta'_4 \circ \Gamma_4 \cup \Gamma'_4$.

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$$\frac{\Delta_1 : A, s \qquad \Gamma_1 : A, s'}{s, s'} \qquad \frac{\Delta_2 : A, s \qquad \Gamma_2 : A, s'}{s, s'}$$

$$\frac{\Delta_3 : A, s \qquad \Gamma_3 : A, s'}{s, s'} \qquad \frac{\Delta_4 : A, s \qquad \Gamma_4 : A, s'}{s, s'}$$

Figure: Cut rules $(C_1), \ldots, (C_4)$.

The introduction rules take their pattern from the introduction rules of the **G3c** sequent system for classical logic.

$$\frac{\Delta_3 : A, s}{\Delta_1 : {\sim_t} A, s} \quad \frac{\Gamma_3 : A, s}{\Gamma_1 : {\sim_t} A, s} \quad \frac{\Delta_4 : A, s}{\Delta_2 : {\sim_t} A, s} \quad \frac{\Gamma_4 : A, s}{\Gamma_2 : {\sim_t} A, s}$$

$$\frac{\Delta_1 : A, s}{\Delta_3 : {\sim_t} A, s} \quad \frac{\Gamma_1 : A, s}{\Gamma_3 : {\sim_t} A, s} \quad \frac{\Delta_2 : A, s}{\Delta_4 : {\sim_t} A, s} \quad \frac{\Gamma_2 : A, s}{\Gamma_4 : {\sim_t} A, s}$$

$$\frac{\Delta_2 : A, s}{\Delta_1 : {\sim_f} A, s} \quad \frac{\Gamma_2 : A, s}{\Gamma_1 : {\sim_f} A, s} \quad \frac{\Delta_1 : A, s}{\Delta_2 : {\sim_f} A, s} \quad \frac{\Gamma_1 : A, s}{\Gamma_2 : {\sim_f} A, s}$$

$$\frac{\Delta_4 : A, s}{\Delta_3 : {\sim_f} A, s} \quad \frac{\Gamma_4 : A, s}{\Gamma_3 : {\sim_f} A, s} \quad \frac{\Delta_3 : A, s}{\Delta_4 : {\sim_f} A, s} \quad \frac{\Gamma_3 : A, s}{\Gamma_4 : {\sim_f} A, s}$$

Figure: Introduction rules $(\sim_t \Delta_1), \ldots, (\sim_t \Gamma_4)$ and $(\sim_f \Delta_1), \ldots, (\sim_f \Gamma_4)$.

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$$\frac{\Delta_1 : A, s}{\Delta_3 : \sim_t A, s} \quad \frac{\Gamma_1 : A, s}{\Gamma_3 : \sim_t A, s} \quad \frac{\Delta_2 : A, s}{\Delta_4 : \sim_t A, s} \quad \frac{\Gamma_2 : A, s}{\Gamma_4 : \sim_t A, s}$$

$$\frac{\Delta_2 : A, s}{\Delta_1 : \sim_f A, s} \quad \frac{\Gamma_2 : A, s}{\Gamma_1 : \sim_f A, s} \quad \frac{\Delta_1 : A, s}{\Delta_2 : \sim_f A, s} \quad \frac{\Gamma_1 : A, s}{\Gamma_2 : \sim_f A, s}$$

$$\frac{\Delta_4 : A, s}{\Delta_3 : \sim_f A, s} \quad \frac{\Gamma_4 : A, s}{\Gamma_3 : \sim_f A, s} \quad \frac{\Delta_3 : A, s}{\Delta_4 : \sim_f A, s} \quad \frac{\Gamma_3 : A, s}{\Gamma_4 : \sim_f A, s}$$

Figure: Introduction rules $(\sim_t \Delta_1), \ldots, (\sim_t \Gamma_4)$ and $(\sim_f \Delta_1), \ldots, (\sim_f \Gamma_4)$.

$$\frac{\Delta_1 : A, s \qquad \Delta_1 : B, s}{\Delta_1 : A \wedge_t B, s} \qquad \frac{\Gamma_1 : A, \Gamma_1 : B, s}{\Gamma_1 : A \wedge_t B, s} \qquad \frac{\Delta_2 : A, s \qquad \Delta_2 : B, s}{\Delta_2 : A \wedge_t B, s} \qquad \frac{\Gamma_2 : A, \Gamma_2 : B, s}{\Gamma_2 : A \wedge_t B, s}$$

$$\frac{\Delta_3 : A, \Delta_3 : B, s}{\Delta_3 : A \wedge_t B, s} \qquad \frac{\Gamma_3 : A, s \qquad \Gamma_3 : B, s}{\Gamma_3 : A \wedge_t B, s} \qquad \frac{\Delta_4 : A, \Delta_4 : B, s}{\Delta_4 : A \wedge_t B, s} \qquad \frac{\Gamma_4 : A, s \qquad \Gamma_4 : B, s}{\Gamma_4 : A \wedge_t B, s}$$

$$\frac{\Delta_1 : A, \Delta_1 : B, s}{\Delta_1 : A \vee_t B, s} \qquad \frac{\Gamma_1 : A, s \qquad \Gamma_1 : B, s}{\Gamma_1 : A \vee_t B, s} \qquad \frac{\Delta_2 : A, \Delta_2 : B, s}{\Delta_2 : A \vee_t B, s} \qquad \frac{\Gamma_2 : A, s \qquad \Gamma_2 : B, s}{\Gamma_2 : A \vee_t B, s}$$

$$\frac{\Delta_3 : A, s \qquad \Delta_3 : B, s}{\Delta_3 : A \vee_t B, s} \qquad \frac{\Gamma_3 : A, \Gamma_3 : B, s}{\Gamma_3 : A \vee_t B, s} \qquad \frac{\Delta_4 : A, s \qquad \Delta_4 : B, s}{\Delta_4 : A \vee_t B, s} \qquad \frac{\Gamma_4 : A, \Gamma_4 : B, s}{\Gamma_4 : A \vee_t B, s}$$

Figure: Introduction rules $(\wedge_t\Delta_1), \ldots, (\wedge_t\Gamma_4)$ and $(\vee_t\Delta_1), \ldots, (\vee_t\Gamma_4)$.

$$\frac{\Delta_1 : A, \Delta_1 : B, s}{\Delta_1 : A \wedge_f B, s} \qquad \frac{\Gamma_1 : A, s \quad \Gamma_1 : B, s}{\Gamma_1 : A \wedge_f B, s} \qquad \frac{\Delta_2 : A, s \quad \Delta_2 : B, s}{\Delta_2 : A \wedge_f B, s} \qquad \frac{\Gamma_2 : A, \Gamma_2 : B, s}{\Gamma_2 : A \wedge_f B, s}$$

$$\frac{\Delta_3 : A, \Delta_3 : B, s}{\Delta_3 : A \wedge_f B, s} \qquad \frac{\Gamma_3 : A, s \quad \Gamma_3 : B, s}{\Gamma_3 : A \wedge_f B, s} \qquad \frac{\Delta_4 : A, s \quad \Delta_4 : B, s}{\Delta_4 : A \wedge_f B, s} \qquad \frac{\Gamma_4 : A, \Gamma_4 : B, s}{\Gamma_4 : A \wedge_f B, s}$$

$$\frac{\Delta_1 : A, s \quad \Delta_1 : B, s}{\Delta_1 : A \vee_f B, s} \qquad \frac{\Gamma_1 : A, \Gamma_1 : B, s}{\Gamma_1 : A \vee_f B, s} \qquad \frac{\Delta_2 : A, \Delta_2 : B, s}{\Delta_2 : A \vee_f B, s} \qquad \frac{\Gamma_2 : A, s \quad \Gamma_2 : B, s}{\Gamma_2 : A \vee_f B, s}$$

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$$\frac{\Delta_1 : B, \Gamma_1 : A, s}{\Delta_1 : A \to_t B, s} \qquad \frac{\Delta_1 : A, s \quad \Gamma_1 : B, s}{\Gamma_1 : A \to_t B, s} \qquad \frac{\Delta_2 : B, \Gamma_2 : A, s}{\Delta_2 : A \to_t B, s} \qquad \frac{\Delta_2 : A, s \quad \Gamma_2 : B, s}{\Gamma_2 : A \to_t B, s}$$

$$\frac{\Delta_3 : B, s \quad \Gamma_3 : A, s}{\Delta_3 : A \to_t B, s} \qquad \frac{\Delta_3 : A, \Gamma_3 : B, s}{\Gamma_3 : A \to_t B, s} \qquad \frac{\Delta_4 : B, s \quad \Gamma_4 : A, s}{\Delta_4 : A \to_t B, s} \qquad \frac{\Delta_4 : A, \Gamma_4 : B, s}{\Gamma_4 : A \to_t B, s}$$

$$\frac{\Gamma_1 : A, s \quad \Delta_1 : B, s}{\Delta_1 : A \to_f B, s} \qquad \frac{\Delta_1 : A, \Gamma_1 : B, s}{\Gamma_1 : A \to_f B, s} \qquad \frac{\Delta_2 : B, \Gamma_2 : A, s}{\Delta_2 : A \to_f B, s} \qquad \frac{\Delta_2 : A, s \quad \Gamma_2 : B, s}{\Gamma_2 : A \to_f B, s}$$

$$\frac{\Gamma_3 : A, s \quad \Delta_3 : B, s}{\Delta_3 : A \to_f B, s} \qquad \frac{\Delta_3 : A, \Gamma_3 : B, s}{\Gamma_3 : A \to_f B, s} \qquad \frac{\Delta_4 : B, \Gamma_4 : A, s}{\Delta_4 : A \to_f B, s} \qquad \frac{\Delta_4 : A, s \quad \Gamma_4 : B, s}{\Gamma_4 : A \to_f B, s}$$

Figure: Introduction rules $(\to_t \Delta_1), \ldots, (\to_t \Gamma_4)$ and $(\to_f \Delta_1), \ldots, (\to_f \Gamma_4)$.

Definition

The system of generalized sequents GL^* is defined as the set consisting of all axiomatic sequents together with the cut and introduction rules. Provability of sequents is defined in the usual way. We say that a formula A is provable in GL^* iff the four sequents $A\circ \mid \circ \mid \circ \mid \circ$, $\circ \mid A\circ \mid \circ \mid \circ$, $\circ \mid \circ \mid \circ A \mid \circ$, and $\circ \mid \circ \mid \circ \mid \circ A$ are provable in GL^*

The rules for $\neg$, $\leftrightarrow_t$, and $\leftrightarrow_f$ are easily obtainable. Nevertheless it might be useful to list them, because they occur in Odintsov's axioms. For $\neg$ we have the rules

$$\frac{\Gamma_i : A, s}{\Delta_i : \neg A, s}\ (\neg\Delta_i), \quad \frac{\Delta_i : A, s}{\Gamma_i : \neg A, s}\ (\neg\Gamma_i).$$

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$$\frac{\Delta_1 : B, \Gamma_1 : A, s \qquad \Delta_1 : A, \Gamma_1 : B, s}{\Delta_1 : A \leftrightarrow_t B, s} \qquad \frac{\Delta_2 : B, \Gamma_2 : A, s \qquad \Delta_2 : A, \Gamma_2 : B, s}{\Delta_2 : A \leftrightarrow_t B, s}$$

$$\frac{\Delta_1 : A, \Delta_1 : B, s \qquad \Delta_1 : A, \Gamma_1 : A, s \qquad \Gamma_1 : B, \Delta_1 : B, s \qquad \Gamma_1 : B, \Gamma_1 : A, s}{\Gamma_1 : A \leftrightarrow_t B, s}$$

$$\frac{\Delta_2 : A, \Delta_2 : B, s \qquad \Delta_2 : A, \Gamma_2 : A, s \qquad \Gamma_2 : B, \Delta_2 : B, s \qquad \Gamma_2 : B, \Gamma_2 : A, s}{\Gamma_2 : A \leftrightarrow_t B, s}$$

$$\frac{\Delta_3 : B, \Gamma_3 : B, s \qquad \Delta_3 : B, \Delta_3 : A, s \qquad \Gamma_3 : A, \Gamma_3 : B, s \qquad \Gamma_3 : A, \Delta_3 : A, s}{\Delta_3 : A \leftrightarrow_t B, s}$$

$$\frac{\Delta_4 : B, \Delta_4 : A, s \qquad \Delta_4 : B, \Gamma_4 : A, s \qquad \Gamma_4 : A, \Delta_4 : B, s \qquad \Gamma_4 : A, \Gamma_4 : Bs}{\Delta_4 : A \leftrightarrow_t B, s}$$

$$\frac{\Delta_3 : A, \Gamma_3 : B, s \qquad \Delta_3 : B, \Gamma_3 : A, s}{\Gamma_3 : A \leftrightarrow_t B, s} \qquad \frac{\Delta_4 : A, \Gamma_4 : B, s \qquad \Delta_4 : B, \Gamma_4 : A, s}{\Gamma_4 : A \leftrightarrow_t B, s}$$

$$\frac{\Gamma_1 : A, s \quad \Gamma_1 : B, s \quad \Gamma_1 : B, s \quad \Delta_1 : A, s}{\Delta_1 : A \leftrightarrow_f B, s} \qquad \frac{\Delta_1 : A, \Gamma_1 : B, \Delta_1 : B, \Gamma_1 : A, s}{\Gamma_1 : A \leftrightarrow_f B, s}$$

$$\frac{\Delta_2 : B, \Gamma_2 : A, \Delta_2 : A, \Gamma_2 : B, s}{\Delta_2 : A \leftrightarrow_f B, s} \qquad \frac{\Delta_2 : A, s \quad \Gamma_2 : B, s \quad \Delta_2 : B, s \quad \Gamma_2 : A, s}{\Gamma_2 : A \leftrightarrow_f B, s}$$

$$\frac{\Gamma_3 : A, s \quad \Delta_3 : B, s \quad \Gamma_3 : B, s \quad \Delta_3 : A, s}{\Delta_3 : A \leftrightarrow_f B, s} \qquad \frac{\Delta_3 : A, \Gamma_3 : B, \Delta_3 : B, \Gamma_3 : A, s}{\Gamma_3 : A \leftrightarrow_f B, s}$$

$$\frac{\Delta_4 : B, \Gamma_4 : A, \Delta_4 : A, \Gamma_4 : B, s}{\Delta_4 : A \leftrightarrow_f B, s} \qquad \frac{\Delta_4 : A, s \quad \Gamma_4 : B, s \quad \Delta_4 : B, s \quad \Gamma_4 : A, s}{\Gamma_4 : A \leftrightarrow_f B, s}$$

Figure: Introduction rules $(\leftrightarrow_t \Delta_1), \ldots, (\leftrightarrow_t \Gamma_4)$ and $(\leftrightarrow_f \Delta_1), \ldots, (\leftrightarrow_f \Gamma_4)$.

Observation

Fore every formula $A \in \mathcal{L}^*_{tf}$, every sequent $\Delta_1 \circ \Gamma_2 \mid \Delta_2 \circ \Gamma_2 \mid \Delta_3 \circ \Gamma_3 \mid \Delta_4 \circ \Gamma_4$, where at least one $\Delta_i \circ \Gamma_i$ $(1 \leq i \leq 4)$ has the shape $\Delta, A \circ \Gamma, A$ is provable in GL^*.

Thus, bottom-up proof search may stop at sequents $\Delta_1 \circ \Gamma_2 \mid \Delta_2 \circ \Gamma_2 \mid \Delta_3 \circ \Gamma_3 \mid \Delta_4 \circ \Gamma_4$, where at least one $\Delta_i \circ \Gamma_i$ $(1 \leq i \leq 4)$ has the shape $\Delta, A \circ \Gamma A$.

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We assume the familiar notions of a derivation and a proof(tree) in sequent calculi, depth of a tree, and depth-preserving admissibility of inference rules.

The depth of a derivation is the maximum length of the branches in the derivation, and the length of a branch is the number of its nodes minus 1.

A rule is admissible in a sequent calculus $\mathcal{C}$ iff (for every instance of the rule) the provability of all the premise sequents in $\mathcal{C}$ guarantees the provability of the conclusion sequent in $\mathcal{C}$. The rule is depth-preserving admissible in $\mathcal{C}$ iff it holds that the conclusion sequent is provable with a prooftree of depth at most n in $\mathcal{C}$, if all the premise sequents are provable with a prooftree of depth at most n in $\mathcal{C}$.

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Observation

The weakening rule

$$\frac{\Delta_1 \circ \Gamma_1 \mid \Delta_2 \circ \Gamma_2 \mid \Delta_3 \circ \Gamma_3 \mid \Delta_4 \circ \Gamma_4}{\Delta_1, \Delta_1' \circ \Gamma_1, \Gamma_1' \mid \Delta_2, \Delta_2' \circ \Gamma_2, \Gamma_2' \mid \Delta_3, \Delta_3' \circ \Gamma_3, \Gamma_3' \mid \Delta_4, \Delta_4' \circ \Gamma_4, \Gamma_4'}$$

is depth-preserving admissible in GL^*.

Observation

The cut rules $(C_1) - (C_8)$ are admissible in cut-free GL^*.

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The weakening rule

$$\frac{\Delta_1 \circ \Gamma_1 \mid \Delta_2 \circ \Gamma_2 \mid \Delta_3 \circ \Gamma_3 \mid \Delta_4 \circ \Gamma_4}{\Delta_1, \Delta_1' \circ \Gamma_1, \Gamma_1' \mid \Delta_2, \Delta_2' \circ \Gamma_2, \Gamma_2' \mid \Delta_3, \Delta_3' \circ \Gamma_3, \Gamma_3' \mid \Delta_4, \Delta_4' \circ \Gamma_4, \Gamma_4'}$$

is depth-preserving admissible in GL^*.

Observation

The cut rules $(C_1) - (C_8)$ are admissible in cut-free GL^*.

Proof. First we can note that the cut-rules $(C_1) - (C_4)$ can be replaced by their context-sharing versions $(C_1)' - (C_4)'$:

$$\frac{\Delta_1 : A, s \quad \Gamma_1 : A, s}{s} \quad \frac{\Delta_2 : A, s \quad \Gamma_2 : A, s}{s}$$

$$\frac{\Delta_3 : A, s \quad \Gamma_3 : A, s}{s} \quad \frac{\Delta_4 : A, s \quad \Gamma_4 : A, s}{s}$$

Thus, it is enough to show that $(C_1)' - (C_4)'$ are admissible in cut-free GL^*. The only difference to the standard proofs of cut-elimination for **G3c** is that the main induction on the cut-rank and the subinduction on the cut-level now is simultaneous for the four rules $(C_1)' - (C_4)'$. ■

Observation

If $\vdash \Delta_1 \circ \Gamma_1 \mid \ldots \mid \Delta_4 \circ \Gamma_4$ *in* GL^**, then for every valuation* v*: there exists* $A \in \Delta_1$ *with* $\mathbf{N} \notin v(A)$ *or there exits* $A \in \Gamma_1$ *with* $\mathbf{N} \in v(A)$ *or there exists* $A \in \Delta_2$ *with* $\mathbf{F} \notin v(A)$ *or there exits* $A \in \Gamma_2$ *with* $\mathbf{F} \in v(A)$ *or there exists* $A \in \Delta_3$ *with* $\mathbf{T} \notin v(A)$ *or there exits* $A \in \Gamma_3$ *with* $\mathbf{T} \in v(A)$ *or there exists* $A \in \Delta_4$ *with* $\mathbf{B} \notin v(A)$ *or there exits* $A \in \Gamma_4$ *with* $\mathbf{B} \in v(A)$*.*

Proposition

If A is provable in GL^*, then $A \in L^{t*}$.

Proof. By contraposition. If it is not the case that $A \in L^{t*}$, then there is a valuation v such that $v(A) \neq \begin{vmatrix} 0 & 0 \\ 1 & 1 \end{vmatrix} = \mathbf{TB}$. If each of the four sequents $A\circ \mid \circ \mid \circ \mid \circ$, $\circ \mid A\circ \mid \circ \mid \circ$, $\circ \mid \circ \mid \circ A \mid \circ$, and $\circ \mid \circ \mid \circ \mid \circ A$ is provable in GL^*, then for every valuation v, $v(A) = \mathbf{TB}$, by the previous Observation. Therefore not all these four sequents are provable in GL^*. ∎

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Proposition

If $A \in L^{t*}$, then A is provable in GL^*.

Proof. By induction of proofs in HL^{t*}, i.e., it has to be shown that (i) *modus ponens* preserves provability in GL^* and (ii) the 38 axioms of HL^{t*} are provable in GL^*. As to (i), assume that the following sequents are provable in GL^*:

$$
\begin{array}{ll}
s_1 = A\circ \mid \circ \mid \circ \mid \circ, & s_2 = \circ \mid A\circ \mid \circ \mid \circ, \\
s_3 = \circ \mid \circ \mid \circ A \mid \circ, & s_4 = \circ \mid \circ \mid \circ \mid \circ A, \\
s'_1 = (A \to_t B)\circ \mid \circ \mid \circ \mid \circ, & s'_2 = \circ \mid (A \to_t B)\circ \mid \circ \mid \circ, \\
s'_3 = \circ \mid \circ \mid \circ(A \to_t B) \mid \circ, & s'_4 = \circ \mid \circ \mid \circ \mid \circ(A \to_t B).
\end{array}
$$

Then the formula B can easily be proved in GL^* using the four cut-rules.

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s_1' = (A \to_t B)\circ \mid \circ \mid \circ \mid \circ, & s_2' = \circ \mid (A \to_t B)\circ \mid \circ \mid \circ, \\
s_3' = \circ \mid \circ \mid \circ(A \to_t B) \mid \circ, & s_4' = \circ \mid \circ \mid \circ \mid \circ(A \to_t B).
\end{array}
$$

Then the formula B can easily be proved in GL^* using the four cut-rules.

Here is one example:

$$\dfrac{s_1' \qquad \dfrac{s_1 \quad \dfrac{A, B \circ A, B \mid \circ \mid \circ \mid \circ}{B \circ A, (A \to_t B) \mid \circ \mid \circ \mid \circ}\,(\to_t \Gamma_1)}{B \circ (A \to_t) \mid \circ \mid \circ \mid \circ}\,(C_1)}{B\circ \mid \circ \mid \circ \mid \circ}\,(C_1)$$

Proving the axioms of HL^{t*} in GL^* is a tedious task, and we shall not present all the cases. Consider, for instance, Axiom 17. We have to derive Axiom 17 in the positions Δ_1, Δ_2, Γ_3 and Γ_4 of otherwise empty sequents. Consider here position Γ_4:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\circ \mid A \circ A \mid \circ \mid \circ}{\circ \sim_f A \mid A\circ \mid \circ \mid \circ}}{\circ \mid A\circ \mid \circ \sim_t \sim_f A \mid \circ}}{\circ \mid A\circ \mid \circ \mid \circ \sim_f \sim_t \sim_f A}}{\circ \mid \circ \mid \circ \mid \sim_t A \circ \sim_f \sim_t \sim_f A} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\circ \mid A \circ A \mid \circ \mid \circ}{\sim_f A\circ \mid \circ A \mid \circ \mid \circ}}{\circ \mid \circ A \mid \sim_t \sim_f A\circ \mid \circ}}{\circ \mid \circ A \mid \circ \mid \sim_f \sim_t \sim_f A\circ}}{\circ \mid \circ \mid \circ \mid \sim_f \sim_t \sim_f A \circ \sim_t A}}{\circ \mid \circ \mid \circ \mid \circ \sim_t A \leftrightarrow_t \sim_f \sim_t \sim_f A}$$

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$$\dfrac{s_1' \qquad \dfrac{s_1 \quad \dfrac{A, B \circ A, B \mid \circ \mid \circ \mid \circ}{B \circ A, (A \to_t B) \mid \circ \mid \circ \mid \circ}\,(\to_t \Gamma_1)}{B \circ (A \to_t) \mid \circ \mid \circ \mid \circ}\,(C_1)}{B\circ \mid \circ \mid \circ \mid \circ}\,(C_1)$$

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The most remarkable axiom is Axiom 38, which we will call Odintsov's axiom, *Od*. To show the provability of *Od* in GL^*, we will introduce some notation:

$$
\begin{aligned}
OdT &= ((A \wedge_t B) \leftrightarrow_t (A \wedge_f B)) \wedge_t ((A \vee_t B) \leftrightarrow_t (A \vee_f B)) \wedge_t ((A \rightarrow_f B) \leftrightarrow_t (A \rightarrow_t B)) \\
OdF &= ((C \wedge_t D) \leftrightarrow_t (C \vee_f D)) \wedge_t ((C \vee_t D) \leftrightarrow_t (C \wedge_f D)) \wedge_t ((C \rightarrow_f D) \leftrightarrow_t (\neg C \wedge_t D)) \\
OdT_1 &= ((A \wedge_t B) \leftrightarrow_t (A \wedge_f B)) \\
OdT_2 &= ((A \vee_t B) \leftrightarrow_t (A \vee_f B)) \\
OdT_3 &= ((A \rightarrow_f B) \leftrightarrow_t (A \rightarrow_t B)) \\
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OdF_3 &= ((C \rightarrow_f D) \leftrightarrow_t (\neg C \wedge_t D))
\end{aligned}
$$

Thus, $Od = OdT \vee_t OdF$, and OdT_i (OdF_i) with $1 \leq i \leq 3$ are the three $\wedge_t$-conjuncts of *OdT* (*OdF*). We here consider *Od* in position Γ_3, split the proof into several parts, and annotate some sequents by names for these sequents.

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Thus, $Od = OdT \vee_t OdF$, and OdT_i (OdF_i) with $1 \leq i \leq 3$ are the three $\wedge_t$-conjuncts of OdT (OdF). We here consider Od in position Γ_3, split the proof into several parts, and annotate some sequents by names for these sequents.

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Here is a simple proof of s_3.

$$\dfrac{\dfrac{\dfrac{\dfrac{\circ \mid \circ \mid B, A \circ B, OdF \mid \circ \qquad \circ \mid \circ \mid A \circ A, B, OdF \mid \circ}{\circ \mid \circ \mid A \to_t B, A \circ B, OdF \mid \circ}}{\circ \mid \circ \mid A \to_t B \circ A \to_f B, OdF \mid \circ} \qquad \dfrac{\dfrac{\dfrac{\circ \mid \circ \mid A \circ A, B, OdF \mid \circ \qquad \circ \mid \circ \mid A, B \circ B, OdF \mid \circ}{\circ \mid \circ \mid A, A \to_f B \circ B, OdF \mid \circ}}{\circ \mid \circ \mid A \to_f B \circ A \to_t B, OdF \mid \circ}}{}}{\circ \mid \circ \mid \circ OdT_3, OdF \mid \circ}$$

In positions Γ_3 and Δ_2 we decompose OdT, and in positions Γ_4 and Δ_1, we decompose OdF.

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In positions Γ_3 and Δ_2 we decompose OdT, and in positions Γ_4 and Δ_1, we decompose OdF.

Theorem

For every formula $A \in \mathcal{L}^*_{tf}$, $\vdash A$ in HL^{t*} iff A is provable in GL^*.

Proposition

Provability of formulas in HL^{t*} is decidable.

The following proposition follows from cut-elimination.

Proposition

Let X be a non-empty subset of $\{\sim_t, \wedge_t, \vee_t, \rightarrow_t, \sim_f, \wedge_f, \vee_f, \rightarrow_f\}$, and let $GL^* \setminus X$ be the result of dropping the introduction rules for the connectives in the complement of X from GL^*. Then a sequent s in the language based on X is provable in GL^* iff it is provable in $GL^* \setminus X$.

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Let X be a non-empty subset of $\{\sim_t, \wedge_t, \vee_t, \rightarrow_t, \sim_f, \wedge_f, \vee_f, \rightarrow_f\}$, and let $GL^* \setminus X$ be the result of dropping the introduction rules for the connectives in the complement of X from GL^*. Then a sequent s in the language based on X is provable in GL^* iff it is provable in $GL^* \setminus X$.

Theorem

For every formula $A \in \mathcal{L}^*_{tf}$, $\vdash A$ in HL^{t*} iff A is provable in GL^*.

Proposition

Provability of formulas in HL^{t*} is decidable.

The following proposition follows from cut-elimination.

Proposition

Let X be a non-empty subset of $\{\sim_t, \wedge_t, \vee_t, \rightarrow_t, \sim_f, \wedge_f, \vee_f, \rightarrow_f\}$, and let $GL^* \setminus X$ be the result of dropping the introduction rules for the connectives in the complement of X from GL^*. Then a sequent s in the language based on X is provable in GL^* iff it is provable in $GL^* \setminus X$.

We conclude this paper by remarking that GL^* provides a *framework* for capturing all the other semantically defined trilattice logics. To deal with the logics L^t and $L^{f'}$ defined with respect to truth entailment, we may just drop the introduction rules for $\rightarrow_f$ and $\rightarrow_t$, respectively. To deal with the logic $\mathcal{L}^{f*}$ defined with respect to falsity entailment, we postulate that a formula A in the full language $\mathcal{L}^*_{tf}$ is provable iff the four sequents

$$\circ A \mid \circ \mid \circ \mid \circ, \quad \circ \mid \circ \mid \circ A \mid \circ, \quad \circ \mid A\circ \mid \circ \mid \circ, \quad \circ \mid \circ \mid \circ \mid A\circ$$

are provable. Sequent systems for fragments of $\mathcal{L}^{f*}$ are obtained by omitting the introduction rules for the connectives in question.

Odintsov makes the following comment:

> *It turns out that the matrix of SIXTEEN$_3$ can be represented as a four-component twist-structure over the two-element Boolean algebra, which suggests also a way of defining an intuitionistic version of the consequence relations $\models_t$ and $\models_f$ via replacing in this twist-structure the two-element Boolean algebra by an arbitrary Heyting algebra.*

The implication $\rightarrow_t$ is Boolean and satisfies the Deduction Theorem. Intuitionistic implicational logic is the minimal pure implicational logic having modus ponens as its only inference rule and satisfying the Deduction Theorem. It is therefore of interest to investigate trilattice logics with intuitionistic implication.

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To obtain intuitionistic trilattice logics, we will replace the metalogical connectives used in Odintsov's matrix presentation by their intuitionistic counterparts.

In the matrix definition of $\sqsupset_f$, for example, $\neg b \wedge b'$ could also be expressed using a symbol for classical co-implication. If we want to have constructive counterparts of the matrix definitions of $\rightarrow_t$ and $\rightarrow_f$, then in addition to intuitionistic implication, we have to consider the constructive co-implication from Heyting-Brouwer logic, alias bi-intuitionistic logic.

Each of the compound formulas that are used in the matrix presentations may be seen as a formula from bi-intuitionistic logic, and for each of these four formulas, we state a pair of tableau rules.

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Each of the compound formulas that are used in the matrix presentations may be seen as a formula from bi-intuitionistic logic, and for each of these four formulas, we state a pair of tableau rules.

We will introduce a tableau calculus IT_{16} in the style of the tableau systems presented in (Priest 2008).

A tableau is a rooted tree. The construction of a tableau starts with an expression $A, 0, \Theta$, where A is a formula from $\mathcal{L}^*_{tf}$, $\Theta \in \{\Delta_i, \Gamma_i\}$, and $1 \leq i \leq 4$. Intuitively, $A, 0, \Delta_i$ can be read as "at state 0, the value of formula A does not contain the element from **4** encoded by i", and $A, 0, \Gamma_i$ can be read as "at state 0, the value of formula A contains the element from **4** encoded by i".

The symbols Δ_i and Γ_i are distinct formal signs generalizing the signs T and F used in tableaux for intuitionistic logic. If j names a state, then A, j, Δ_i and A, j, Γ_i is a pair of contradictory expressions, the appearance of which closes a tableau branch.

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To an expression $A, 0, \Theta$ decomposition rules and structural tableau rules may (or may not) be applied to obtain a tableau. Some decomposition rules introduce expressions of the form jrk. Intuitively, jrk provides semantical information about an intuitionistic Kripke frame. It indicates that the information state k is a possible expansion of the information state j.

The indices j, k, l are (names of) elements from $\mathbb{N}$, and a new index is the (name of the) smallest natural number not already used in the tableau. If an expression is placed below an arrow, it is a rule output, otherwise it is a rule input.

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The indices j, k, l are (names of) elements from $\mathbb{N}$, and a new index is the (name of the) smallest natural number not already used in the tableau. If an expression is placed below an arrow, it is a rule output, otherwise it is a rule input.

The structural rule ref for which no input is stated may be applied to any node to introduce an expression *jrj* if *j* occurs on the tableau.

Some decomposition rules introduce a new information state.

If an expression of the form *jrk* (*krj*) is a rule input on a tableau branch, the rule has to be re-applied after the introduction of an expression *jrl* (*lrj*) not already on the branch. If a rule application yields an expression already on the branch, the rule is not applied.

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If an expression of the form jrk (krj) is a rule input on a tableau branch, the rule has to be re-applied after the introduction of an expression jrl (lrj) not already on the branch. If a rule application yields an expression already on the branch, the rule is not applied.

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Definition

A tableau branch is said to be closed iff there are expressions of the form A, k, Δ_i and A, k, Γ_i on the branch. A tableau is called closed iff all of its branches are closed. If a tableau (tableau branch) is not closed, it is called *open*. A tableau branch is said to be complete iff no more rules can be applied to expand it. A tableau is said to be complete iff each of its branches is complete.

ref	tran	pers
$\cdot$	jrk	p, j, Γ_i
$\downarrow$	krl	jrk
krk	$\downarrow$	$\downarrow$
	jrl	p, k, Γ_i
for all k on the tableau		$1 \leq i \leq 4$

Table: Structural tableau rules and persistence rules

Definition

The tableau calculus IT_{16} consists of the structural and persistence rules from Table 4 and the following decomposition rules:

$$\begin{array}{c} \sim_t A, j, \Delta_1 \\ \downarrow \\ A, j, \Delta_3 \end{array} \qquad \begin{array}{c} \sim_t A, j, \Gamma_1 \\ \downarrow \\ A, j, \Gamma_3 \end{array} \qquad \begin{array}{c} \sim_t A, j, \Delta_2 \\ \downarrow \\ A, j, \Delta_4 \end{array} \qquad \begin{array}{c} \sim_t A, j, \Gamma_2 \\ \downarrow \\ A, j, \Gamma_4 \end{array}$$

$$\begin{array}{c} \sim_t A, j, \Delta_3 \\ \downarrow \\ A, j, \Delta_1 \end{array} \qquad \begin{array}{c} \sim_t A, j, \Gamma_3 \\ \downarrow \\ A, j, \Gamma_1 \end{array} \qquad \begin{array}{c} \sim_t A, j, \Delta_4 \\ \downarrow \\ A, j, \Delta_2 \end{array} \qquad \begin{array}{c} \sim_t A, j, \Gamma_4 \\ \downarrow \\ A, j, \Gamma_2 \end{array}$$

$$\begin{array}{c} A \vee_t B, j, \Delta_1 \\ \swarrow \quad \searrow \\ A, j, \Delta_1 \quad B, j, \Delta_1 \end{array} \qquad \begin{array}{c} A \vee_t B, j, \Gamma_1 \\ \downarrow \\ A, j, \Gamma_1 \\ B, j, \Gamma_1 \end{array} \qquad \begin{array}{c} A \vee_t B, j, \Delta_2 \\ \swarrow \quad \searrow \\ A, j, \Delta_2 \quad B, j, \Delta_2 \end{array} \qquad \begin{array}{c} A \vee_t B, j, \Gamma_2 \\ \downarrow \\ A, j, \Gamma_2 \\ B, j, \Gamma_2 \end{array}$$

$$\begin{array}{c} A \vee_t B, j, \Delta_3 \\ \downarrow \\ A, j, \Delta_3 \\ B, j, \Delta_3 \end{array} \qquad \begin{array}{c} A \vee_t B, j, \Gamma_3 \\ \swarrow \quad \searrow \\ A, j, \Gamma_3 \quad B, j, \Gamma_3 \end{array} \qquad \begin{array}{c} A \vee_t B, j, \Delta_4 \\ \downarrow \\ A, j, \Delta_4 \\ B, j, \Delta_4 \end{array} \qquad \begin{array}{c} A \vee_t B, j, \Gamma_4 \\ \swarrow \quad \searrow \\ A, j, \Gamma_4 \quad B, j, \Gamma_4 \end{array}$$

Definition

$$\begin{array}{c} A \wedge_t B, j, \Delta_1 \\ \downarrow \\ A, j, \Delta_1 \\ B, j, \Delta_1 \end{array}$$

$$\begin{array}{c} A \wedge_t B, j, \Gamma_1 \\ \swarrow \quad \searrow \\ A, j, \Gamma_1 \quad B, j, \Gamma_1 \end{array}$$

$$\begin{array}{c} A \wedge_t B, j, \Delta_2 \\ \downarrow \\ A, j, \Delta_2 \\ B, j, \Delta_2 \end{array}$$

$$\begin{array}{c} A \wedge_t B, j, \Gamma_2 \\ \swarrow \quad \searrow \\ A, j, \Gamma_2 \quad B, j, \Gamma_2 \end{array}$$

$$\begin{array}{c} A \wedge_t B, j, \Delta_3 \\ \swarrow \quad \searrow \\ A, j, \Delta_3 \quad B, j, \Delta_3 \end{array}$$

$$\begin{array}{c} A \wedge_t B, j, \Gamma_3 \\ \downarrow \\ A, j, \Gamma_3 \\ B, j, \Gamma_3 \end{array}$$

$$\begin{array}{c} A \wedge_t B, j, \Delta_4 \\ \swarrow \quad \searrow \\ A, j, \Delta_4 \quad B, j, \Delta_4 \end{array}$$

$$\begin{array}{c} A \wedge_t B, j, \Gamma_4 \\ \downarrow \\ A, j, \Gamma_4 \\ B, j, \Gamma_4 \end{array}$$

$$\begin{array}{c} A \rightarrow_t B, j, \Delta_1 \\ krj \\ \swarrow \quad \searrow \\ A, k, \Gamma_1 \quad B, k, \Delta_1 \end{array}$$

$$\begin{array}{c} A \rightarrow_t B, j, \Gamma_1 \\ \downarrow \\ krj \\ A, k, \Delta_1 \\ B, k, \Gamma_1 \end{array}$$

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$$\begin{array}{c} A \rightarrow_t B, j, \Gamma_2 \\ \downarrow \\ krj \\ A, k, \Delta_2 \\ B, k, \Gamma_2 \end{array}$$

$$\begin{array}{c} A \rightarrow_t B, j, \Delta_3 \\ \downarrow \\ jrk \\ A, k, \Gamma_3 \\ B, k, \Delta_3 \end{array}$$

$$\begin{array}{c} A \rightarrow_t B, j, \Gamma_3 \\ jrk \\ \swarrow \quad \searrow \\ A, k, \Delta_3 \quad B, k, \Gamma_3 \end{array}$$

$$\begin{array}{c} A \rightarrow_t B, j, \Delta_4 \\ \downarrow \\ jrk \\ A, k, \Gamma_4 \\ B, k, \Delta_4 \end{array}$$

$$\begin{array}{c} A \rightarrow_t B, j, \Gamma_4 \\ jrk \\ \swarrow \quad \searrow \\ A, k, \Delta_4 \quad B, k, \Gamma_4 \end{array}$$

Definition

$$\begin{array}{c} \sim_f A, j, \Delta_1 \\ \downarrow \\ A, j, \Delta_2 \end{array} \qquad \begin{array}{c} \sim_f A, j, \Gamma_1 \\ \downarrow \\ A, j, \Gamma_2 \end{array} \qquad \begin{array}{c} \sim_f A, j, \Delta_2 \\ \downarrow \\ A, j, \Delta_1 \end{array} \qquad \begin{array}{c} \sim_f A, j, \Gamma_2 \\ \downarrow \\ A, j, \Gamma_1 \end{array}$$

$$\begin{array}{c} \sim_f A, j, \Delta_3 \\ \downarrow \\ A, j, \Delta_4 \end{array} \qquad \begin{array}{c} \sim_f A, j, \Gamma_3 \\ \downarrow \\ A, j, \Gamma_4 \end{array} \qquad \begin{array}{c} \sim_f A, j, \Delta_4 \\ \downarrow \\ A, j, \Delta_3 \end{array} \qquad \begin{array}{c} \sim_f A, j, \Gamma_4 \\ \downarrow \\ A, j, \Gamma_3 \end{array}$$

$$\begin{array}{c} A \vee_f B, j, \Delta_1 \\ \downarrow \\ A, j, \Delta_1 \\ B, j, \Delta_1 \end{array} \qquad \begin{array}{c} A \vee_f B, j, \Gamma_1 \\ \swarrow \quad \searrow \\ A, j, \Gamma_1 \quad B, j, \Gamma_1 \end{array} \qquad \begin{array}{c} A \vee_f B, j, \Delta_2 \\ \swarrow \quad \searrow \\ A, j, \Delta_2 \quad B, j, \Delta_2 \end{array} \qquad \begin{array}{c} A \vee_f B, j, \Gamma_2 \\ \downarrow \\ A, j, \Gamma_2 \\ B, j, \Gamma_2 \end{array}$$

$$\begin{array}{c} A \vee_f B, j, \Delta_3 \\ \downarrow \\ A, j, \Delta_3 \\ B, j, \Delta_3 \end{array} \qquad \begin{array}{c} A \vee_f B, j, \Gamma_3 \\ \swarrow \quad \searrow \\ A, j, \Gamma_3 \quad B, j, \Gamma_3 \end{array} \qquad \begin{array}{c} A \vee_f B, j, \Delta_4 \\ \swarrow \quad \searrow \\ A, j, \Delta_4 \quad B, j, \Delta_4 \end{array} \qquad \begin{array}{c} A \vee_f B, j, \Gamma_4 \\ \downarrow \\ A, j, \Gamma_4 \\ B, j, \Gamma_4 \end{array}$$

Definition

$$\begin{array}{c} A \wedge_f B, j, \Delta_1 \\ \swarrow \quad \searrow \\ A, j, \Delta_1 \quad B, j, \Delta_1 \end{array} \qquad \begin{array}{c} A \wedge_f B, j, \Gamma_1 \\ \downarrow \\ A, j, \Gamma_1 \\ B, j, \Gamma_1 \end{array} \qquad \begin{array}{c} A \wedge_f B, j, \Delta_2 \\ \downarrow \\ A, j, \Delta_2 \\ B, j, \Delta_2 \end{array} \qquad \begin{array}{c} A \wedge_f B, j, \Gamma_2 \\ \swarrow \quad \searrow \\ A, j, \Gamma_2 \quad B, j, \Gamma_2 \end{array}$$

$$\begin{array}{c} A \wedge_f B, j, \Delta_3 \\ \swarrow \quad \searrow \\ A, j, \Delta_3 \quad B, j, \Delta_3 \end{array} \qquad \begin{array}{c} A \wedge_f B, j, \Gamma_3 \\ \downarrow \\ A, j, \Gamma_3 \\ B, j, \Gamma_3 \end{array} \qquad \begin{array}{c} A \wedge_f B, j, \Delta_4 \\ \downarrow \\ A, j, \Delta_4 \\ B, j, \Delta_4 \end{array} \qquad \begin{array}{c} A \wedge_f B, j, \Gamma_4 \\ \swarrow \quad \searrow \\ A, j, \Gamma_4 \quad B, j, \Gamma_4 \end{array}$$

$$\begin{array}{c} A \rightarrow_f B, j, \Delta_1 \\ \downarrow \\ jrk \\ A, k, \Gamma_1 \\ B, k, \Delta_1 \end{array} \qquad \begin{array}{c} A \rightarrow_f B, j, \Gamma_1 \\ jrk \\ \swarrow \quad \searrow \\ A, k, \Delta_1 \quad B, k, \Gamma_1 \end{array} \qquad \begin{array}{c} A \rightarrow_f B, j, \Delta_2 \\ krj \\ \swarrow \quad \searrow \\ A, k, \Gamma_2 \quad B, k, \Delta_2 \end{array} \qquad \begin{array}{c} A \rightarrow_f B, j, \Gamma_2 \\ \downarrow \\ krj \\ A, k, \Delta_2 \\ B, k, \Gamma_2 \end{array}$$

$$\begin{array}{c} A \rightarrow_f B, j, \Delta_3 \\ \downarrow \\ jrk \\ A, k, \Gamma_3 \\ B, k, \Delta_3 \end{array} \qquad \begin{array}{c} A \rightarrow_f B, j, \Gamma_3 \\ jrk \\ \swarrow \quad \searrow \\ A, k, \Delta_3 \quad B, k, \Gamma_3 \end{array} \qquad \begin{array}{c} A \rightarrow_f B, j, \Delta_4 \\ krj \\ \swarrow \quad \searrow \\ A, k, \Gamma_4 \quad B, k, \Delta_4 \end{array} \qquad \begin{array}{c} A \rightarrow_f B, j, \Gamma_4 \\ \downarrow \\ krj \\ A, k, \Delta_4 \\ B, k, \Gamma_4 \end{array}$$

If k occurs below an arrow, then it is a *new* index.

Example

The following tableau, where p is an atomic formula, has an infinite branch indicated by vertical dots:

$$((p \to_f (p \to_f p)) \to_f (p \to_f p)) \to_f p, 0, \Delta_1$$

$$\downarrow$$

$$\begin{array}{c} \text{0r0} \\ \text{0r1} \\ (p \to_f (p \to_f p)) \to_f (p \to_f p), 1, \Gamma_1 \\ p, 1, \Delta_1 \end{array}$$

$$\downarrow$$

$$\text{1r1}$$

$$\swarrow \qquad\qquad \searrow$$

$$(p \to_f (p \to_f p)), 1, \Delta_1 \qquad \ldots$$

$$\downarrow$$

$$\begin{array}{c} \text{1r2} \\ p, 2, \Gamma_1 \\ p \to_f p, 2, \Delta_1 \end{array}$$

$$\swarrow \qquad\qquad \searrow$$

$$(p \to_f (p \to_f p)), 2, \Delta_1 \qquad \ldots$$

$$\downarrow$$

$$\begin{array}{c} \text{2r3} \\ p, 3, \Gamma_1 \\ p \to_f p, 3, \Delta_1 \end{array}$$

$$\vdots$$

Definition

An $\mathcal{L}^*_{tf}$-formula A is provable in position Δ_i $(1 \leq i \leq 4)$ iff there is a closed complete tableau for $A, 0, \Gamma_i$. The formula A is provable in position Γ_i $(1 \leq i \leq 4)$ iff there is a closed complete tableau for $A, 0, \Delta_i$.

Definition

An $\mathcal{L}^*_{tf}$-formula A is provable in the tableau calculus IT_{16} iff A is provable in positions Δ_1, Δ_2, Γ_3, and Γ_4.

Thus, a formula A is not provable iff A is not provable in position Δ_1 or it is not provable in position Δ_2 or it is not provable in position Γ_3 or it is not provable in Γ_4. This is the case iff there is an open branch in every complete tableau for $A, 0, \Gamma_1$ or an open branch in every complete tableau for $A, 0, \Gamma_2$ or an open branch in every complete tableau for $A, 0, \Delta_3$ or an open branch in every complete tableau for $A, 0, \Delta_4$. If we want to prove a formula in IT_{16} we need not, of course, expand a closed branch.

Example

A proof of a formula consists of four closed tableau. Let p and q be any propositional variables and consider $p \to_t (q \to_t p)$. The provability of this formula in positions Δ_1 and Δ_2 can be shown by completely analogous closed tableaux and so can the provability in positions Γ_3 and Γ_4. We show only one tableau for provability in position Δ_1 and one for provability in position Γ_3. In both tableaux the final rule applied is pers.

$$
\begin{array}{c}
p \to_t (q \to_t p), 0, \Gamma_1 \\
\downarrow \\
1r0 \\
p, 1, \Delta_1 \\
q \to_t p, 1, \Gamma_1 \\
\downarrow \\
2r1 \\
q, 2, \Delta_1 \\
p, 2, \Gamma_1 \\
\downarrow \\
p, 1, \Gamma_1
\end{array}
\qquad
\begin{array}{c}
p \to_t (q \to_t p), 0, \Delta_3 \\
\downarrow \\
0r1 \\
p, 1, \Gamma_1 \\
q \to_t p, 1, \Delta_1 \\
\downarrow \\
1r2 \\
q, 2, \Gamma_1 \\
p, 2, \Delta_1 \\
\downarrow \\
p, 2, \Gamma_1
\end{array}
$$

We define a relational semantics with four kinds of evaluation relations.

Definition

A Kripke model is a structure $\mathcal{M} = \langle \mathcal{F}, \models_1, \models_2, \models_3, \models_4, v \rangle$, where $\mathcal{F} = \langle M, R \rangle$ is a Kripke frame and v is a function from $Atom \times M$ into $\mathbf{16}$ satisfying the following persistence condition: for all $x, y \in M$ and $\sharp \in \mathbf{4}$, if xRy and $\sharp \in v(p, x)$, then $\sharp \in v(p, y)$. For $\mathcal{M} = \langle \langle M, R \rangle, \models_1, \models_2, \models_3, \models_4, v \rangle$, the relations $\models_i \subseteq (M \times \mathcal{L}^*_{tf})$ $(1 \leq i \leq 4)$ are defined as follows:

$$
\begin{array}{lll}
x \models_1 p & \text{iff} & \mathbf{N} \in v(p, x), \\
x \models_2 p & \text{iff} & \mathbf{F} \in v(p, x), \\
x \models_3 p & \text{iff} & \mathbf{T} \in v(p, x), \\
x \models_4 p & \text{iff} & \mathbf{B} \in v(p, x),
\end{array}
$$

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x \models_3 p & \text{iff} & \mathbf{T} \in v(p, x), \\
x \models_4 p & \text{iff} & \mathbf{B} \in v(p, x),
\end{array}
$$

Definition

$$
\begin{aligned}
x \models_1 \sim_t A \quad &\text{iff} \quad x \models_3 A,\\
x \models_2 \sim_t A \quad &\text{iff} \quad x \models_4 A,\\
x \models_3 \sim_t A \quad &\text{iff} \quad x \models_1 A,\\
x \models_4 \sim_t A \quad &\text{iff} \quad x \models_2 A,
\end{aligned}
$$

$$
\begin{aligned}
x \models_1 A \wedge_t B \quad &\text{iff} \quad x \models_1 A \textit{ or } x \models_1 B,\\
x \models_2 A \wedge_t B \quad &\text{iff} \quad x \models_2 A \textit{ or } x \models_2 B,\\
x \models_3 A \wedge_t B \quad &\text{iff} \quad x \models_3 A \textit{ and } x \models_3 B,\\
x \models_4 A \wedge_t B \quad &\text{iff} \quad x \models_4 A \textit{ and } x \models_4 B,
\end{aligned}
$$

Definition

$$
\begin{array}{lll}
x \models_1 A \vee_t B & \text{iff} & x \models_1 A \textit{ and } x \models_1 B, \\
x \models_2 A \vee_t B & \text{iff} & x \models_2 A \textit{ and } x \models_2 B, \\
x \models_3 A \vee_t B & \text{iff} & x \models_3 A \textit{ or } x \models_3 B, \\
x \models_4 A \vee_t B & \text{iff} & x \models_4 A \textit{ or } x \models_4 B,
\end{array}
$$

$$
\begin{array}{lll}
x \models_1 A \to_t B & \text{iff} & (\exists y \in M)\, yRx \textit{ and } y \not\models_1 A \textit{ and } y \models_1 B, \\
x \models_2 A \to_t B & \text{iff} & (\exists y \in M)\, yRx \textit{ and } y \not\models_2 A \textit{ and } y \models_2 B, \\
x \models_3 A \to_t B & \text{iff} & (\forall y \in M)\, xRy \textit{ implies } (y \not\models_3 A \textit{ or } y \models_3 B), \\
x \models_4 A \to_t B & \text{iff} & (\forall y \in M)\, xRy \textit{ implies } (y \not\models_4 A \textit{ or } y \models_4 B),
\end{array}
$$

Definition

$$
\begin{aligned}
x \models_1 \sim_f A \quad &\text{iff} \quad x \models_2 A,\\
x \models_2 \sim_f A \quad &\text{iff} \quad x \models_1 A,\\
x \models_3 \sim_f A \quad &\text{iff} \quad x \models_4 A,\\
x \models_4 \sim_f A \quad &\text{iff} \quad x \models_3 A,
\end{aligned}
$$

$$
\begin{aligned}
x \models_1 A \wedge_f B \quad &\text{iff} \quad x \models_1 A \textit{ and } x \models_1 B,\\
x \models_2 A \wedge_f B \quad &\text{iff} \quad x \models_2 A \textit{ or } x \models_2 B,\\
x \models_3 A \wedge_f B \quad &\text{iff} \quad x \models_3 A \textit{ and } x \models_3 B,\\
x \models_4 A \wedge_f B \quad &\text{iff} \quad x \models_4 A \textit{ or } x \models_4 B,
\end{aligned}
$$

Definition

$$
\begin{array}{lll}
x \models_1 A \vee_f B & \text{iff} & x \models_1 A \textit{ or } x \models_1 B, \\
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x \models_3 A \vee_f B & \text{iff} & x \models_3 A \textit{ or } x \models_3 B, \\
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\end{array}
$$

$$
\begin{array}{lll}
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x \models_3 A \to_f B & \text{iff} & (\forall y \in M)\, xRy \textit{ implies } (y \not\models_3 A \textit{ or } y \models_3 B), \\
x \models_4 A \to_f B & \text{iff} & (\exists y \in M)\, yRx \textit{ and } y \not\models_4 A \textit{ and } y \models_4 B.
\end{array}
$$

Proposition

Let $\mathcal{M} = \langle \mathcal{F}, \models_1, \models_2, \models_3, \models_4, v \rangle$ be a Kripke model and A be an $\mathcal{L}^*_{tf}$-formula. Then for all $x, y \in M$ and $i \in \{1, 2, 3, 4\}$, if xRy and $x \models_i A$, then $y \models_i A$.

Definition

Suppose that $\mathcal{M} = \langle \langle M, R \rangle, \models_1, \models_2, \models_3, \models_4, v \rangle$ is a Kripke model. An $\mathcal{L}^*_{tf}$-formula A is IT_{16}-valid in $\mathcal{M}$ iff for every $x \in M$: $x \not\models_1 A$, $x \not\models_2 A$, $x \models_3 A$, $x \models_4 A$. An $\mathcal{L}^*_{tf}$-formula A is IT_{16}-valid iff it is IT_{16}-valid in every Kripke model.

Proposition

Let $\mathcal{M} = \langle \mathcal{F}, \models_1, \models_2, \models_3, \models_4, v \rangle$ be a Kripke model and A be an $\mathcal{L}^*_{tf}$-formula. Then for all $x, y \in M$ and $i \in \{1, 2, 3, 4\}$, if xRy and $x \models_i A$, then $y \models_i A$.

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Suppose that $\mathcal{M} = \langle \langle M, R \rangle, \models_1, \models_2, \models_3, \models_4, v \rangle$ is a Kripke model. An $\mathcal{L}^*_{tf}$-formula A is IT_{16}-valid in $\mathcal{M}$ iff for every $x \in M$: $x \not\models_1 A$, $x \not\models_2 A$, $x \models_3 A$, $x \models_4 A$. An $\mathcal{L}^*_{tf}$-formula A is IT_{16}-valid iff it is IT_{16}-valid in every Kripke model.

Example

Here is a simple example of a complete and open tableau.

$$
\begin{array}{c}
p \vee_t (p \to_t q), 0, \Delta_4 \\
\downarrow \\
p, 0, \Delta_4 \\
p \to_t q, 0, \Delta_4 \\
\downarrow \\
0\mathrm{r}1 \\
p, 1, \Gamma_4 \\
q, 1, \Delta_4 \\
\downarrow \\
0\mathrm{r}0 \\
1\mathrm{r}1 \\
\downarrow \\
p, 0, \Gamma_4 \\
q, 0, \Delta_4
\end{array}
$$

From this tableau one can easily read off a model based on the set of indices $\{0, 1\}$ with $0 \not\models_4 p$, $0 \not\models_4 q$, $1 \models_4 p$, and $1 \not\models_4 q$. Thus, $0 \not\models_4 p \vee_t (p \to_t q)$ and therefore $p \vee_t (p \to_t q)$ fails to be IT_{16}-valid.

Definition

Let $\mathcal{M} = \langle\langle M, R\rangle, \models_1, \models_2, \models_3, \models_4, v\rangle$ be a Kripke model, and *br* be a tableau branch. The model $\mathcal{M}$ is faithful to *br* iff there exists a function f from $\mathbb{N}$ into M such that:

- for every expression A, j, Γ_i on *br*, $f(j) \models_i A$;
- for every expression A, j, Δ_i on *br*, $f(j) \not\models_i A$;
- for every expression jrk on *br*, $f(j)Rf(k)$.

The function f is said to *show that $\mathcal{M}$ is faithful to branch br*.

Lemma

Let $\mathcal{M} = \langle\langle M, R\rangle, \models_1, \models_2, \models_3, \models_4), v\rangle$ be a Kripke model, and br be a tableau branch. If $\mathcal{M}$ is faithful to br, and a tableau rule is applied to br, then the application produces at least one extension br′ of br, such that $\mathcal{M}$ is faithful to br′.

Theorem

(Soundness) If an $\mathcal{L}^*_{tf}$-formula A is s provable in IT_{16}, then A is IT_{16}-valid.

Corollary

The rule PERS

$$\begin{array}{c} A, j, \Gamma_i \\ j r k \\ \downarrow \\ A, k, \Gamma_i \end{array}$$

$$1 \leq i \leq 4$$

is admissible for arbitrary $\mathcal{L}^*_{tf}$-formulas A.

Lemma

Let $\mathcal{M} = \langle\langle M, R\rangle, \models_1, \models_2, \models_3, \models_4), v\rangle$ be a Kripke model, and br be a tableau branch. If $\mathcal{M}$ is faithful to br, and a tableau rule is applied to br, then the application produces at least one extension br′ of br, such that $\mathcal{M}$ is faithful to br′.

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Let $\mathcal{M} = \langle\langle M, R\rangle, \models_1, \models_2, \models_3, \models_4), v\rangle$ be a Kripke model, and br be a tableau branch. If $\mathcal{M}$ is faithful to br, and a tableau rule is applied to br, then the application produces at least one extension br′ of br, such that $\mathcal{M}$ is faithful to br′.

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is admissible for arbitrary $\mathcal{L}^*_{tf}$-formulas A.

We now prove completeness of the tableau calculus IT_{16} with respect to the class of all Kripke models.

Definition

Let br be an open branch of a complete tableau. The model $\mathcal{M}_{br} = \langle\langle M_{br}, R_{br}\rangle, \models_1, \models_2, \models_3, \models_4, v_{br}\rangle$ induced by br is defined as follows:

- $M_{br} := \{w_j \mid j \text{ occurs on } br\}$,
- $w_j R_{br} w_k$ iff jrk occurs on br,
- $\mathbf{N} \in v_{br}(p, w_j)$ iff p, j, Γ_1 occurs on br,
- $\mathbf{F} \in v_{br}(p, w_j)$ iff p, j, Γ_2 occurs on br,
- $\mathbf{T} \in v_{br}(p, w_j)$ iff p, j, Γ_3 occurs on br,
- $\mathbf{B} \in v_{br}(p, w_j)$ iff p, j, Γ_4 occurs on br.

The model $\mathcal{M}_{br}$ is well-defined, because *br* is open. There exists no $w_j \in M_{br}$, no atomic formula p, and no $\sharp \in \{\mathbf{N}, \mathbf{F}, \mathbf{T}, \mathbf{B}\}$ such that $\sharp \in v_{br}(p, w_j)$ and $\sharp \notin v_{br}(p, w_j)$. Moreover, the relation R_{br} is reflexive and transitive and the persistence condition is satisfied, because *br* is complete and hence the rules ref, tran, and pers have been applied.

Lemma

Suppose that br is an open branch of a complete tableau, and let $\mathcal{M}_{br} = \langle\langle M_{br}, R_{br}\rangle, \models_1, \models_2, \models_3, \models_4, v_{br}\rangle$ *be the model induced by br. Then for every* $i \in \{1, 2, 3, 4\}$*:*

- *if* A, j, Γ_i *occurs on br, then* $w_j \models_i A$*,*
- *if* A, j, Δ_i *occurs on br, then* $w_j \not\models_i A$*.*

Theorem

(Completeness) Let A be an $\mathcal{L}^*_{tf}$-formula. If A is IT_{16}-valid, then A is provable in IT_{16}.

Proof.

Suppose that A is not provable in IT_{16}. Then there is an open branch br_1 of a complete tableau for $A, 0, \Gamma_1$ or an open branch br_2 of a complete tableau for $A, 0, \Gamma_2$ or an open branch br_3 of a complete tableau for $A, 0, \Delta_3$ or an open branch br_4 of a complete tableau for $A, 0, \Delta_4$. By the previous lemma, $w_0 \models_1 A$ or $w_0 \models_2 A$ or $w_0 \not\models_3 A$ or $w_0 \not\models_4 A$. In any case A is not valid in $\mathcal{M}_{br_i}$ for some $i \in \{1, 2, 3, 4\}$ and hence A is not IT_{16}-valid. $\square$

Corollary

If one complete tableau for $A, 0, \Gamma_1$ is closed, one complete tableau for $A, 0, \Gamma_2$ is closed, one complete tableau for $A, 0, \Delta_3$ is closed, and one complete tableau for $A, 0, \Delta_4$ is closed, then every complete tableau for $A, 0, \Gamma_1$ is closed, every complete tableau for $A, 0, \Gamma_2$ is closed, every complete tableau for $A, 0, \Delta_3$ is closed, and every complete tableau for $A, 0, \Delta_4$ is closed.

The calculus IT_{16} is an intuitionistic tableau-counterpart of the sequent calculus GL^* for truth entailment in $SIXTEEN_3$. A sequent calculus for falsity entailment in $SIXTEEN_3$ can be obtained form GL^* by suitably modifying the notion of provability of an $\mathcal{L}^*_{tf}$-formula in the sequent calculus. In a similar way, the notion of provability in IT_{16} can be modified.

Definition

The tableau calculus IF_{16} is the same as IT_{16}. An $\mathcal{L}^*_{tf}$-formula A is provable in IF_{16} iff A is provable in positions Γ_1, Δ_2, Γ_3, and Δ_4.

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